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An Exact Solution for the Dispersive Alfven Switch-on Shock

Robert L. Hamilton
George Fox University, rhamilto@georgefox.edu

C C. Wu
University of California - Los Angeles

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R. L. Hamilton

Department of Physics, Angelo State University, San Angelo, Texas, USA

C. C. Wu

Department of Physics and Astronomy, University of California, Los Angeles, Los Angeles, California, USA

[1] An exact solution for the dispersive Alfven switch-on shock is found in the case of weak nonlinearity, weak dispersion, and weak damping. Two singular and apparently nonphysical solutions are found as well. The switch-on shock solution is shown to reduce, in the appropriate limits, to two documented special cases of the shock: one case being a description of the planar shock without dispersion and the other being an asymptotic description of the circularly polarized wave standing upstream from the shock in the dispersive case. The solution developed here provides the complete shock structure for either case. This exact solution may serve as a convenient basis for an analytical study of the stability of dispersive switch-on shocks. *INDEX TERMS:* 7851 Space Plasma Physics: Shock waves; 7839 Space Plasma Physics: Nonlinear phenomena; 3210 Mathematical Geophysics: Modeling; 2752 Magnetospheric Physics: MHD waves and instabilities; *KEYWORDS:* nonlinear, dispersive, Alfven, switch-on, shock, bowshock

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1. Introduction

[2] Alfven waves have been observed in a variety of settings [*Burlaga*, 1983; *Hoppe et al.*, 1981; *Kennel et al.*, 1984; *Tsurutani and Smith*, 1986]. Observations of Alfven switch-on shocks, on the other hand, are apparently confined to a single reported detection of a quasi-parallel shock near the Earth's bow shock [*Farris et al.*, 1994]. This scarcity of observations of the field-aligned switch-on shock near the bow shock is attributed, in part, to the relative geometry of the bow shock and the solar wind [*Farris et al.*, 1994; *Greenstadt*, 1991]. A further constraint is provided by the limited parameter regime in which switch-on shocks may form [*De Sterck and Poedts*, 1999a], specifically that the plasma β and Alfvenic Mach number, M_A, must satisfy the condition,

$$1 < M_A < \sqrt{\frac{\gamma(1-\beta)+1}{\gamma-1}}. \tag{1}$$

It should be noted that the numerical simulations of *De Sterck and Poedts* [1999a] of a two-dimensional (2-D) axisymmetrical flow do show the formation of switch-on shocks along with significant structural changes of the modeled bow shock correlated with this switch-on parameter regime.

[3] The conventional starting point for studies of the nonlinear behavior of Alfven waves is found in the one-dimensional magnetohydrodynamic (MHD) Navier-Stokes equations [*Landau et al.*, 1984]. The reduction of this full set of equations for parallel, or quasi-parallel, waves in the limit of weak nonlinearity, weak dispersion, and weak resistive damping leads to the so-called Derivative Nonlinear Schrodinger-Burgers (DNLSB) equation [*Wyller and Mjolhus*, 1984]

$$\frac{\partial B}{\partial t} + \alpha \frac{\partial}{\partial x}\left(|B^2|B\right) = (\bar{R} - iR)\frac{\partial^2 B}{\partial x^2}, \tag{2}$$

where $B(x, t) = B_y + iB_z$. Here B_y, B_z, α, $\bar{R}$, and R are real valued with B_y and B_z being the components of the magnetic field transverse to a uniform, ambient magnetic field pointing in the x-direction. The value of α in the nonlinear term is given by [*Kennel and Edmiston*, 1988]

$$\alpha = \frac{1}{4}\frac{C_I^2}{C_I^2 - C_S^2}, \tag{3}$$

where C_I and C_S are the intermediate Alfven and sound speeds upstream. It is important to observe that the validity of the model equation requires that C_I and C_S are not too close. As noted by *De Sterck and Poedts* [1999a], the switch-on parameter regime requires $\beta < 2/\gamma$ or, in other words, that $(C_S)^2 < (C_I)^2$. Therefore the DNLSB description of the shock will not be valid in the high β end of the switch-on parameter regime which, by equation (1) above or Figure 3 of *De Sterck and Poedts* [1999a], is also the narrowest part of the allowed parameter regime with the Alfvenic Mach number constrained close to unity. Finally, $\bar{R}$ and R give the strength of the damping and dispersion respectively.

[4] Without damping ($\bar{R}$ = 0 with R $\neq$ 0), the resulting form of the DNLSB equation is integrable and has a rich complement of exact solutions [*Kawata and Inoue*, 1978; *Hamilton et al.*, 1992; *Mjolhus and Hada*, 1997]. Without dispersion, (R = 0, with $\bar{R} \neq 0$), the resulting equation also has a known solution [*Kennel and Edmiston*, 1988]. The DNLSB equation has been used to model the effects of dissipation on DNLS solitons [*Wyller and Mjolhus*, 1984; *Hada et al.*, 1993], but no exact shock solution for the DNLSB has been published to date. The exact solution for the dispersive Alfven switch-on shock presented here is shown to reduce to an asymptotic form for the dispersive switch-on shock found by *Coroniti* [1971] as well as the nondispersive switch-on shock reported by *Kennel and Edmiston* [1988]. It should be noted that *Osin* [1989] has reported a rather remarkable exact solution of the full set of MHD equations, including Hall dispersion, which also was shown to reduce to the asymptotic form found by *Coroniti* [1971]. While Osin's result is exact and is not limited by a small amplitude constraint, it is an implicit solution and so might be less tractable in analytical applications.

[5] The exact solution for the weakly dispersive and weakly nonlinear Alfven switch-on shock presented here may be of use in further analytical or numerical studies of this shock. While the applicability of this shock solution is constrained by the limits of the model equation discussed above, the same equation without dispersion [*Kennel et al.*, 1990; *Wu and Kennel*, 1992] has been demonstrated [*De Stercks and Poedts*, 1999a, 1999b] to be a valuable model in illuminating the properties of small amplitude MHD shocks, most notably with regard to the intermediate shock waves. One salient application of an exact solution would be in a study of the conditions of stability of the shock. While the switch-on shocks observed in the Earth's bow shock [*Farris et al.*, 1994] and in the numerical model of the bow shock [*De Sterck and Poedts*, 1999a] were evidently stable under the conditions of their observation, it is not clear, for example, how they would be effected by interaction with finite amplitude waves traveling parallel to the field or at an angle to it [*Kennel and Edmiston*, 1988]. While analytical arguments regarding stability can be made in the absence of an exact solution [*Hada*, 1994], knowledge of an exact solution has allowed for a more detailed understanding of the stability of waves in some cases [*Mjolhus*, 1976; *Laedke and Spatschek*, 1982; *Terasawa et al.*, 1986; *Ruderman*, 1987; *Mjolhus and Hada*, 1990; *Khabibrakhmanov and Summers*, 1995] and the solution presented here is a natural starting point for a comparable analytical study of the switch-on shock.

2. Derivation of Exact Solutions to the DNLSB

2.1. Weiss, Tabor, and Carnevale Method

[6] The solution method used in this paper is based on work by Weiss, Tabor, and Carnevale (WTC) [*Weiss et al.*, 1983] who proposed a generalized Painleve analysis appropriate for the direct study of nonlinear partial differential equations. The basis for this approach is an assumed expansion for the solution of the form

$$u(x,t) = \phi(x,t)^{\varepsilon} \sum_{j=0}^{\infty} \phi(x,t)^{j},$$

where $\phi(x, t)$ is the singular manifold for the solution u(x, t). It has been demonstrated [*Clarkson and Cosgrove*, 1987; *Cariello and Tabor*, 1989] that useful solutions may be found from self-consistent, truncated expansions of this form even for nonlinear partial differential equations not integrable by techniques such as the Inverse Scattering Transformation [*Ablowitz and Segur*, 1985].

[7] As the DNLSB is a complex valued equation it is convenient to study instead the pair of coupled equations

$$u_t + \alpha \frac{\partial}{\partial x}\left(u^2 v\right) = r u_{xx} \tag{4a}$$

$$v_t + \alpha \frac{\partial}{\partial x}\left(v^2 u\right) = \tilde{r} v_{xx}, \tag{4b}$$

where $r = \bar{R} - iR$. This set is equivalent to equation (1) with $u = v^*$. The truncation attempted is simply

$$u(x,t) = a(x,t)\phi(x,t)^{\varepsilon} \tag{5a}$$

$$v(x,t) = b(x,t)\phi(x,t)^{\eta}. \tag{5b}$$

Upon placing this singular expansion in equation (2), the existence of a solution requires the coefficients of the various powers of ϕ(x,t) to sum to zero. Balancing the most singular of the resulting terms constrains ε and η to be

$$\varepsilon = -1/2 - i\mu, \quad \eta = -1/2 + i\mu, \quad \text{with } \mu = \frac{R}{2\bar{R}}. \tag{6}$$

Similarly, the conditions imposed on a(x, t), b(x, t) and ϕ (x, t) at the various orders are

$$ab = K\phi_x \text{ where } K = -\frac{r\tilde{r}}{2\nu\alpha} = -\frac{R^2 + \bar{R}^2}{2\bar{R}\alpha} \tag{7}$$

$$\varepsilon a\phi_t + \alpha\left(a^2 b\right)_x = r\varepsilon[2a_x\phi_x + a\phi]_{xx} \tag{8a}$$

$$\eta b\phi_t + \alpha\left(b^2 a\right)_x = \tilde{r}\varepsilon[2b_x\phi_x + b\phi]_{xx} \tag{8b}$$

$$a_t = r a_{xx} \tag{9a}$$

$$b_t = \tilde{r} b_{xx}. \tag{9b}$$

Equation (7) can be used to simplify equations (8a) and (8b) to yield

$$\phi_t = r\phi_x \frac{\partial}{\partial x}\ln(a) = \tilde{r}\phi_x \frac{\partial}{\partial x}\ln(b). \tag{10}$$

Once again, using equation (7), this can be written as

$$\phi_t = -\alpha K \phi_{xx}. \tag{11}$$

Substituting this expression for ϕ_t into equation (10) and integrating gives a(x,t) and b(x,t) in terms of ϕ_x:

$$a(x,t) = f(t)(\phi_x)^{-\varepsilon}, \quad b(x,t) = g(t)(\phi_x)^{-\eta}, \tag{12}$$

where $f(t)g(t) = K$

Placing these expressions for a(x,t) and b(x,t) into equations (9a) and (9b) yields

$$\frac{\partial}{\partial t}\ln(f(t)) = \frac{r^2\tilde{r}}{4\bar{R}}\left[\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - \frac{\phi_{xxx}}{\phi_x}\right] = 0. \tag{13}$$

Upon integrating equation (13) and ensuring compatibility with equation (11), it is found that

$$\phi(x,t) = c_2\exp[c_1(x - \alpha K c_1 t)] + c_3, \tag{14}$$

where c_1, c_2 and c_3 are real constants.

2.2. An Exact Dispersive Switch-On Shock Solution and Limiting Forms

[8] Finally, from equations (5a)(5b), (12), and (14), an exact solution for the DNLSB (2) may be written as:

$$B(x,t) = B_0\left[1 + e^{2\left(x-\alpha B_0^2 t\right)/\bar{L}}\right]^{-\frac{\left(1+i\frac{R}{\bar{R}}\right)}{2}}. \tag{15}$$

Far upstream, the field approaches the form (at t = 0)

$$B(x,t) \cong B_0 e^{-x/\bar{L}} e^{-ix\frac{R}{\bar{R}}/\bar{L}} \text{ for } x \to +\infty, \tag{16}$$

which matches an asymptotic solution of the switch-on shock found by *Coroniti* [1971]. In this region, the shock width, $\bar{L}$, can be defined by

$$\bar{L} = \frac{R^2 + \bar{R}^2}{\bar{R}\alpha B_0^2} \tag{17a}$$

and the wavelength of the oscillatory, upstream section of the shock is given by

$$L = \frac{R^2 + \bar{R}^2}{R\alpha B_0^2}, \tag{17b}$$

where B_0 is the transverse component of the magnetic field in the downstream state. It can be seen that the number of oscillations in the circularly polarized standing wave upstream is proportional to the ratio of L to $\bar{L}$ or, more simply, $R/\bar{R}$. The accompanying figure shows plots of B_y, B_z and $|B|^2$ as a function of position at time t = 0 along with a hodogram of B_y and B_z.

[9] In the dispersion free limit, B(x,t) can be written as (at t = 0)

$$B(x,t)^2 = \left[1 + e^{2x/\bar{L}}\right]^{-1}, \tag{18}$$

with, in this case, $\bar{L} = \frac{\bar{R}}{\alpha B_0^2}$. This matches a solution found by *Kennel and Edmiston* [1988].

2.3. Singular Solutions of the DNLSB

[10] Last, two apparently nonphysical, singular solutions are reported

$$B(x,t) = B_0\left[1 - e^{2\left(x-\alpha B_0^2 t\right)/\bar{L}}\right]^{-\frac{\left(1+i\frac{R}{\bar{R}}\right)}{2}} \quad \text{for } (x - \alpha B_0 t) < 0 \tag{19a}$$

$$B(x,t) = B_0\left[e^{-2\left(x+\alpha B_0^2 t\right)/\bar{L}} - 1\right]^{-\frac{\left(1+i\frac{R}{\bar{R}}\right)}{2}} \quad \text{for } (x + \alpha B_0 t) < 0. \tag{19b}$$

The divergence of the singular solutions in equation (19a) and (19b) violates the small amplitude assumptions under which the DNLSB equation was derived [*Wyller and Mjolhus*, 1984]. As such, it is inferred that they are nonphysical solutions.

3. Summary and Discussion

[11] It is not evident how to extend the WTC method to find exact solutions for other Alfven shocks. It has been used successfully to find solutions to nonlinear model equations for a variety of physical systems, for example, by *Clarkson and Cosgrove* [1987] and *Cariello and Tabor* [1989], and it seems likely to be of use in space physics applications in addition to the weakly nonlinear, weakly dispersive switch-on shock found here.

[12] The exact solution for equation (15) for the dispersive Alfven switch-on shock is shown to reduce to two previously known limiting cases [*Coroniti*, 1971; *Kennel and Edmiston*, 1988]. The model equation on which this solution is based can be derived from the one-dimensional, dissipative MHD equations including Hall dispersion in the limit of weak nonlinearity, weak dispersion, and wave propagation at small angles to the upstream magnetic field [*Wyller and Mjolhus*, 1984]. It does describe the mode coupling between the intermediate and fast Alfven waves, but not mode coupling with the sound wave, so that for $\beta \sim 2/\gamma$ the two-mode description of the equation breaks down. This means that the switch-on shock solution presented above should be valid through most of the parameter regime in which switch-on shocks are allowed with the exception of the higher β end at $\beta < 2/\gamma$. Last, the source of dissipation used in the model equation may be suspect in applications to the solar wind.

[13] Within these limitations, the conditions for which the shock structure was well understood were the cases without dispersion [*Kennel and Edmiston*, 1988], resulting in a planar shock, or an asymptotic solution of the shock including the effects of dispersion [*Coroniti*, 1971], describing a circularly polarized wave standing upstream from the shock. As shown above and in Figure 1, the solution presented here provides the complete shock structure either with or without dispersion.

[14] The switch-on shock has been observed in the Earth's bow shock [*Farris et al.*, 1994] and in numerical simulations of the bow shock [*De Sterck and Poedts*, 1999a]. As discussed by *Farris et al.* [1994], it seems likely that the presence of this shock is the exception for the Earth's bow shock. It is also the case that given a field-aligned flow [*De Sterck and Poedts*, 1999a], that the switch-on shock is one fundamental means available to the flow, in the switch-on shock parameter regime of equation (1), to adapt topologically to the presence of an object in its path. The observations of these authors provide a motivation to develop a better understanding of the physics of the switch-on shock and, for example, to clarify its behavior and stability in the presence of other finite amplitude waves, turbulence, or 3-D perturbations. For an analytical study of these properties there are many methods

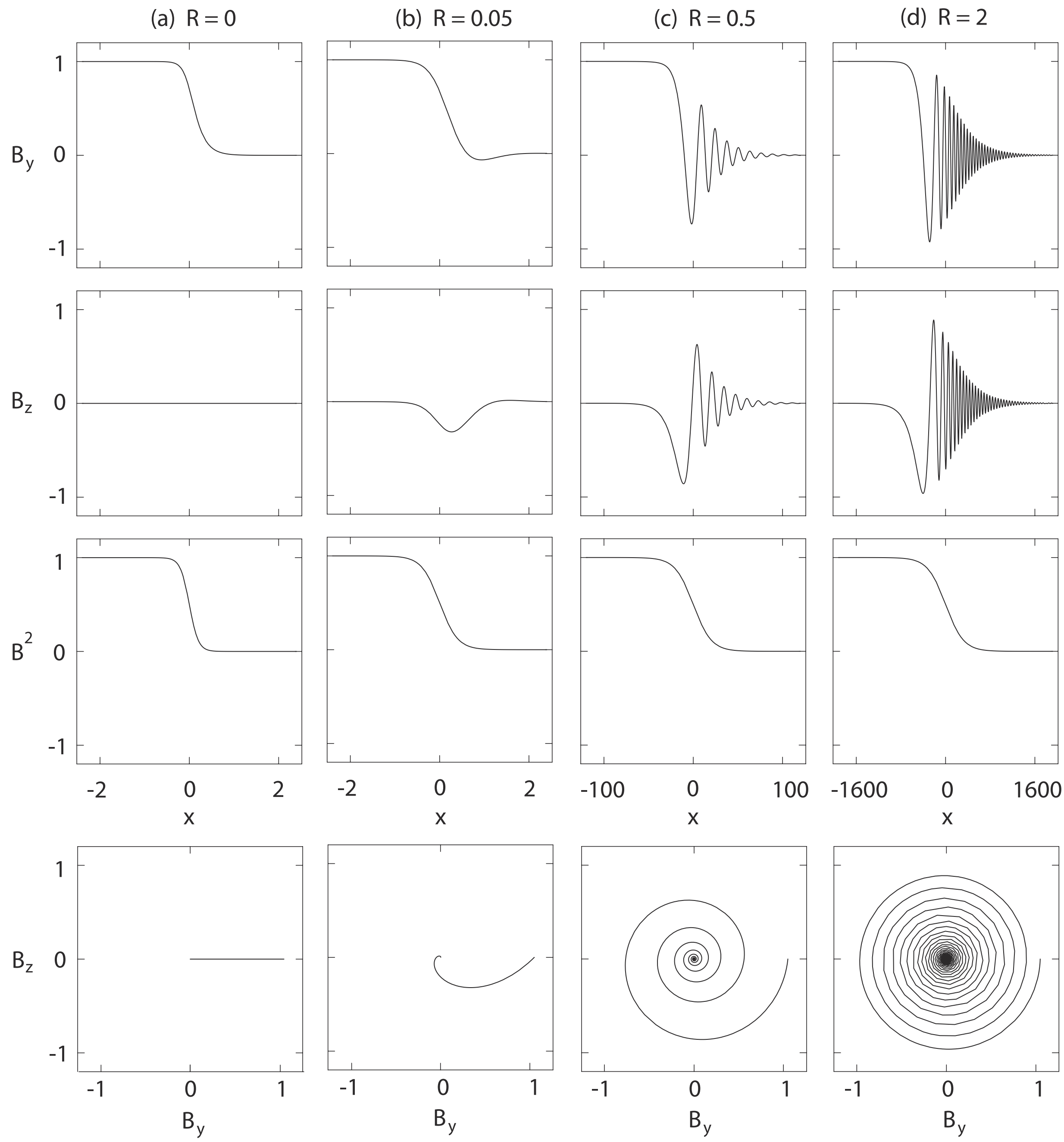

Figure 1. The values used for dispersion, R, are shown at the top of each column. The value for dissipation is taken to be 0.05. Other values are $\beta = 0$, $\alpha = 1/4$, and Bo = 1. Note that the range of the plots increases to the right as the value of R increases. The damping wavelengths, $\bar{L}$, are 0.2, 0.4, 20.2, and 320.2 for Figures 1a–1d, respectively. (a) Without dispersion, the shock is planar. With relatively strong dispersion compared to damping, $R/\bar{R} \gg 1$ as in Figures 1c and 1d, the shock has a circularly polarized wave train standing upstream from it. The exact solution provides the complete shock structure for these limiting cases as well as for intermediate values of dispersion as in Figure 1b.

available, some examples of which are cited above. A natural starting point for such an analytical study is the weakly dispersive, weakly nonlinear switch-on shock solution presented here.

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R. L. Hamilton, Department of Physics, Angelo State University, San Angelo, Texas, USA.

C. C. Wu, Department of Physics and Astronomy, University of California, Los Angeles, Los Angeles, California, USA.