1993

Alfvén Solitons and the DNLS Equation

Robert L. Hamilton
George Fox University, rhamilto@georgefox.edu

C F. Kennel
University of California - Santa Barbara

E Mjølhus
University of Tromso

Alfvén Solitons and the DNLS Equation

R.L. Hamilton[1,*], *C.F. Kennel*[1,*], and *E. Mjølhus*[2]

[1]Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

[2]Institute for Mathematical and Physical Sciences, University of Tromsø, Tromsø, Norway

[*]Permanent address: Institute for Geophysics and Planetary Physics, Department of Physics, University of California, Los Angeles, USA

There is a significant body of literature related to the analytic modeling of Alfven waves in the solar wind which takes dispersive magnetohydrodynamics as an idealized basis. In this context, the derivative nonlinear Schrodinger (DNLS) equation has been found by several authors [1-5] to describe the evolution of small amplitude Alfven waves. It may be scaled to the form

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}(|b|^2 b) + i\frac{\partial^2 b}{\partial x^2} = 0 \tag{1}$$

where b(x,t) is the complex representation of the magnetic field perpendicular to the direction of propagation, x. Although the DNLS neglects a rich variety of mechanisms which affect the propagation of Alfven waves in the solar wind, it does provide a powerful tool for studying their underlying nonlinear behavior.

The value of the DNLS as a theoretical basis for studying Alfven waves is in large part due to its integrability through the inverse scattering transformation (IST). Kaup and Newell [6] found the appropriate Lax pair and developed the IST for so-called vanishing boundary conditions with $b(x) \to 0$ for $x \to \pm\infty$. These boundary conditions are appropriate for a localized perturbation travelling parallel to a static magnetic field. Kawata and Inoue [7] developed the IST with non-vanishing boundary conditions appropriate for oblique Alfven waves and Kawata et al. [8] went on to treat the case of a localized perturbation travelling on a circularly polarized carrier wave. Lastly, Prikarpatskii [9] has dealt with the DNLS under periodic boundary conditions.

While for applications to space physics the IST is a particularly promising approach to the DNLS, the equation itself is amenable to other forms of integrability tests. For example, Mikhailov et al. [10] list it amongst a class of equations which pass a test for integrability based on a symmetry approach. By writing two Hamiltonian decompositions of the DNLS, we show here that the DNLS also satisfies a beautiful formalism for integrable systems set forth by Magri [11]:

$$b_t = M\, Q_1(b) = L\, Q_2(b)$$

$$M = i\,\partial_x^2 + \partial_x[\, b\, \partial_x^{-1}\, (\, \bar{b}\, \partial_x + b\, \overline{\partial_x}\,)] \qquad Q_1(b) = b$$

where

$$L = \partial_x \qquad Q_2(b) = b^2 \bar{b} + i b_x$$

$$\partial_x^{-1} \equiv \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) dx$$

and M and L are symplectic operators with respect to the bilinear form:

$$\langle f, g \rangle = \int_{+\infty}^{-\infty} [\bar{f} g + f \bar{g}]\, dx$$

This immediately leads to the recursion operator, $L^{-1}M$, for the infinite sequence of conservation laws of the DNLS.

An advantage of the IST, over other techniques for the analytic study of the DNLS, is that it allows one to consider the relation of initial conditions to the formation of solitons. This has been done by Ichikawa and Abe [12], Mjolhus [13] and Dawson and Fontan [14] for the case of parallel Alfven waves. Numerical studies of soliton formation have also been carried out for this case by a number of authors [15-19]. In these works, the modulational instability, discussed in an early paper by Mjolhus [20], appears as a useful criterion for the formation of Alfven solitons.

The importance of non-vanishing boundary conditions for space physics applications has been stressed [21,22] due to the possible existence of oblique Alfven waves or the possible refraction of initially parallel Alfven waves [23]. As noted by Kawata and Inoue [7], under non-vanishing boundary conditions the DNLS has a two-parameter family of solitons corresponding to a set of discrete complex eigenvalues of the scattering problem and a one-parameter family whose solitons are either bright or dark and correspond to a set of discrete real eigenvalues. Of these soliton families, only the two-parameter family continues to exist for parallel propagation.

The existence of the one-parameter family is connected to the unique physical setting of oblique Alfven wave propagation [5,24,25]. To sketch this setting, we note that the DNLS describes the mode coupling between the Alfven and magnetosonic waves. For strictly parallel propagation, the underlying MHD phase speeds of these two waves coincide. As the angle of propagation is increased, however, the underlying wave speeds will separate and thus gradually diminish the role of mode coupling in the nonlinear development of these waves. Even so, the DNLS provides an accurate description of the Alfven and magnetosonic waves in this oblique regime. In fact, depending on the choice of waveframe, the DNLS has been shown [5,24] to reduce to either the KdV description of the magnetosonic wave [26] or the MKdV description of the Alfven wave [27]. Moreover, the velocity of a one-parameter DNLS soliton is not only bounded by the underlying MHD magnetosonic and Alfven wave speeds, but as its speed approaches either the magnetosonic or the Alfven speed, it will deform into a KdV or MKdV soliton respectively [5].

The formation of Alfven solitons under non-vanishing boundary conditions has been considered by Hamilton, Kennel and Mjolhus [25] through an analytic study of the scattering data for a set of initial field profiles. It is found that one-parameter solitons are formed in trains which can unambiguously be identified as either Alfvenic (with the slowest soliton speed approaching the Alfven speed) or magnetosonic (with the fastest soliton speed approaching the magnetosonic speed). The solitons in a given train are either all dark (rarefactive) or all bright (compressive) and, if both an Alfvenic and magnetosonic train are formed from a given initial profile, then the fastest soliton in the Alfvenic train must be slower than any soliton of the magnetosonic train. It was found, though, that the eigenvalues of the fastest Alfvenic soliton and the slowest magnetosonic soliton may coalesce if the wavenumber of the initial profile is large enough make a section of the profile modulationally unstable. The coalescence of two real eigenvalues forms a degenerate and structurally unstable soliton which will bifurcate into a two-parameter soliton as the wavenumber of the initial profile is increased. More detailed observations of the formation of Alfven solitons are contained in reference [25]. The Gelfand-Levitan equations for the degenerate one-parameter soliton have been solved in reference [28].

Acknowledgments

This research was supported by NASA under grant No. NASAW-1624. One of the authors, RLH, would like to acknowledge and thank the Institute of Mathematical and Physical Sciences of the University of Tromso for support and hospitality during his stay there. While at the Institute for Theoretical Physics, RLH was supported in part by the National Science Foundation under grant No. PHY 89-04035. RLH would like to thank Dr. Y. H. Ichikawa and Dr. T. Hada for their comments.

References

1. Rogister, A., Phys. Fluids **14**, 2733 (1971).
2. Mjolhus, E., Report no. 48, Department of Applied Mathematics, University of Bergen, Norway (1974).
3. Mjolhus, E., Wyller, J., J. Plasma Physics **40**, 299 (1988).
4. Ruderman, M. S., Fluid Dyn. 22, 299 (1987), (Izv. Akad. Nauk SSSR, Mekh. Zhid. i Gasa, 2, 159 (1987)).
5. Kennel, C. F., Buti, B., Hada, T., Pellat, R., Phys. Fluids **31**, 1949 (1988).
6. Kaup, D. J., Newell, A. C., J. Math. Phys. **19**, 798 (1978).
7. Kawata, T., Inoue, H., J. Phys. Soc. Japan **44**, 1968 (1978).
8. Kawata, T., Kobayashi, N., Inoue, H., J. Phys. Soc. Japan 46, 1008 (1979).
9. Prikarpatskii, A. K., Theor. Mat. Phys. **47**, 487 (1981).
10. Mikhailov, A. V., Shabat, A. B, Sokolov, V. V., in "What is Integrability?", V. E. Zacharov (Ed.), Springer-Verlag (1991).
11. Magri, F., J. Math. Phys. **19**, 1156 (1977).
12. Ichikawa, Y. H., Abe, Y., Prog. Theor. Phys. Supp., No. 94, 128 (1988).
13. Mjolhus, E., J. Plasma Physics **19**, 437 (1978).
14. Dawson, S. P., Fontan, C. F., Phys. Rev. A, 39, 5289 (1989).
15. Spangler, S. R., Sheerin, J. P., Payne, G. L., Phys. Fluids **28**, 104 (1985).
16. Spangler, S. R., Astrophys. J. **299**, 122 (1985).
17. Ghosh, S., Papadopoulos, K.,Phys. Fluids **30**, 1371 (1987).
18. Dawson, S. P., Fontan, C. F., Phys. Fluids **31**, 83 (1988).
19. Dawson, S. P., "Solitons and Radiation from the Derivative nonlinear Schrodinger Equation", preprint 1990.
20. Mjolhus, E., J. Plasma Phys. **16**, 321 (1976).
21. Kennel, C. F., Buti, B., Hada, T., Pellat, R., Phys. Fluids **31**, 1949 (1988).
22. Mjolhus, E., Wyller, J., Physica Scripta **33**, 442 (1985).
23. Hada, T., Kennel, C. F., Terasawa, T., J. Geophys. Res. **88**, 4423 (1987).
24. Mjolhus, E., Physica Scripta **40**, 227 (1989).

25. Hamilton, R. L., Kennel, C. F., Mjolhus, E., Institute for Theoretical Physics Preprint No. NSF-ITP-91-131.
26. Kakutani, T., Ono, H. , Taniuti, T., Wei, C.-C., J. Phys. Soc. Japan **24**, 1159 (1968).
27. Kakutani, T., Ono, H. , J. Phys. Soc. Japan **26**, 1305 (1969).
28. Hamilton, R. L., Kennel, C. F., Mjolhus, E., Institute for Theoretical Physics Preprint No. NSF-ITP-91-130.