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An Investigation of Subtraction Algorithms from the 18th and 19th Centuries

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An Investigation of Subtraction Algorithms from the 18th and 19th Centuries - Textbooks and Cyphering Books

An Investigation of Subtraction Algorithms from the 18th and 19th Centuries - Definitions and Algorithms

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Introduction

How do you subtract? What is your algorithm of choice? The most common subtraction algorithm in the United States today is the *decomposition algorithm*. While other algorithms exist and are possibly utilized, no other algorithm dominates the contemporary curriculum, classrooms, and pencils of children in the U.S. more so than the decomposition algorithm. Subtraction is one of the fundamental building blocks of arithmetic. Looking to the past and exploring the history of subtraction algorithms can help us glean knowledge of the intended and possibly implemented curricula in past eras. In this paper, special attention will be given to 18th and 19th century America. The aims of this paper are to identify the different algorithms used during this time period and to discuss implications for the modern teacher.

Textbooks and Cyphering Books

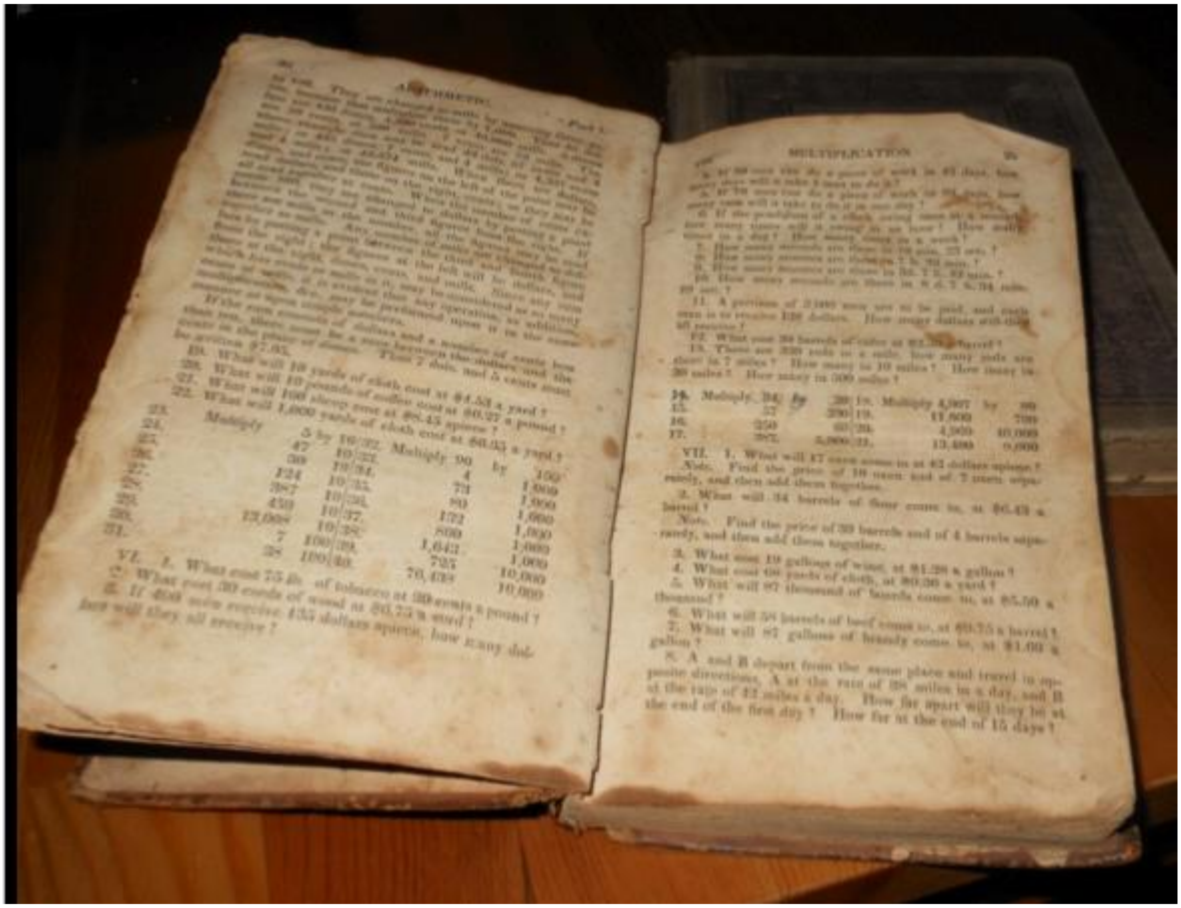


Figure 1. An example of an arithmetic text, *First lessons in arithmetic* (2nd ed.) by Warren Colburn, published in 1824 in Boston, Massachusetts, by Hilliard & Metcalf. (Photo used courtesy of the book owner, Nicole M. Wessman-Enzinger.)

Utilizing printed books as the solitary source in this investigation would be an insufficient representation of this time period. This is because, as recently shown by Nerida Ellerton and Ken Clements in their book, *Rewriting the history of school mathematics in North America 1607-1861* (2012), in North America during the 18th and 19th centuries, printed arithmetic texts were used less often than handwritten cyphering books. Cyphering books, an integral part of the "abbaco" tradition from Europe, were handwritten accounts of some branch of mathematics, usually arithmetic or geometry, prepared by and belonging to specific individuals. In fact, these books were often referred to by those who created them as "my book" or, for example, as "John Smith, his book." They were typically prepared by students attending school or college or an evening class, both children and adults, and were utilized by their owners throughout their lifetimes. The entries are all in handwritten ink, usually very neat, and some books contain illustrations. The spelling "cyphering" versus the modern "ciphering" is an intentional semantic choice. The word "cyphering" was commonly used before 1800, and the term "copybook" may have been used as well (Ellerton & Clements, 2012).

Cyphering books were copied from a teacher's or sometimes a parent's own cyphering books and contain mathematical definitions, rules, and examples (e.g., Brayton, 1792; Butlar, 1785; Green, 1795-1800; Hall, 1780; Moses, 1801; Prust, 1702). As such, these cyphering books represent a valuable look into the *implemented* curriculum of the era. The printed arithmetic books, representative of the intended curriculum, are what we would now call "textbooks." In this article, the use of words such as "books" or "arithmetic books" will usually refer to printed arithmetic textbooks. The data for this research came from both of these sources, the printed arithmetic books and the handwritten cyphering books, to represent the implemented and intended curriculum of the time.

For data collection, arithmetic books were selected from those in Nerida Ellerton's and Ken Clements' large collection of printed books and were also accessed through Google Books. Approximately 32 arithmetic books were examined from these sources. Ellerton and Clements have established the nation's largest and most representative collection of cyphering books. All 280 of the cyphering books then in the Ellerton-Clements collection were examined in 2012. The arithmetic and cyphering books from both Google Books and the Ellerton-Clements collection are considered to be a representative sample of the books used in 18th century Colonial America and in the United States during the 18th and 19th centuries.

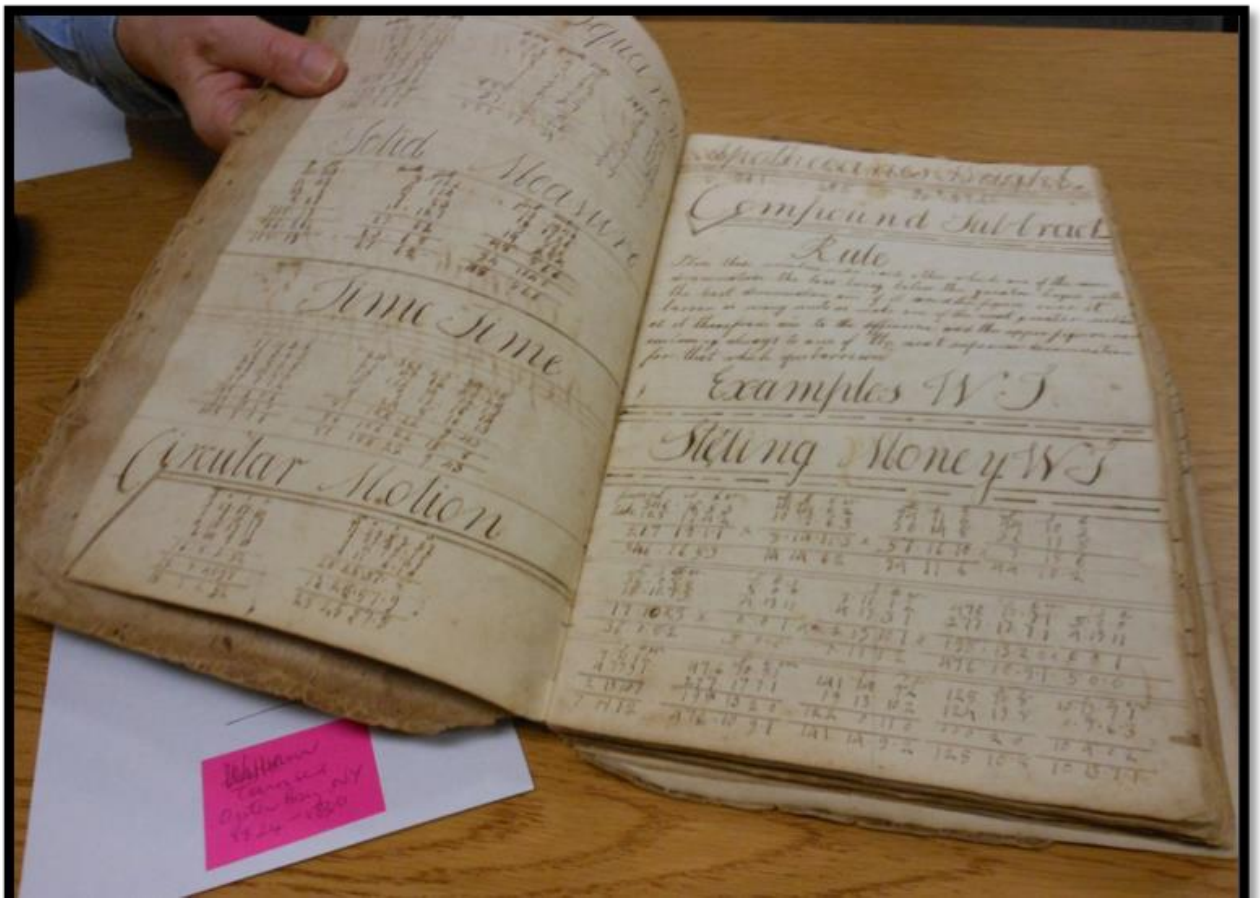


Figure 2. An example of a cyphering book (W. Townsend, 1824-1830, New York) from the Ellerton-Clements collection. (Photo used with the permission of the book owners, Nerida F. Ellerton and M. A. (Ken) Clements.)

Note on source: The images of pages of cyphering books in this article have been reproduced, with permission, from cyphering books belonging to Nerida F. Ellerton and M. A. (Ken) Clements, who, as of January 2014, own a collection of 350 North American cyphering books, dating from 1701 to 1861. This is the largest collection, public or private, of North American cyphering books. For details relating to these cyphering books, consult *Rewriting the History of School Mathematics in North America 1607–1861,* by N. F. Ellerton and M. A. Clements (Springer, 2012). Ellerton and Clements have indicated that photographs of excerpts from the cyphering books in their collection can be reproduced in scholarly papers or presentations or in curriculum materials, provided it is indicated that the cyphering books from which the photographs were taken are in the Ellerton-Clements collection and provided reference is made to the above-mentioned book by Ellerton and Clements (2012).

Definition of Subtraction

The definitions of subtraction presented in the arithmetic books and cyphering books from 1700 to 1900 are not the same as those given in our modern classrooms and textbooks because they exclude the possibility of negative numbers.Edward Cocker (1702, p. 29) wrote,

Subtraction is the taking of a less Number out of a greater of like kind, whereby to find out a third Number, being or declaring the inequality, excess, or difference between the numbers given.

Similarly, John Ayres (1711, p. 33) wrote,

Subtraction teacheth to take the lesser number from a greater, or an equal from an equal; whereby we discover the remainder excess, or difference.

The phrase a “lesser number from a greater number” is present in nearly all of the books during this time period (e.g., Brookes, 1776; Dilworth, 1810; Adams, 1848). The definitions of subtraction from 1700 to 1900 are stated in a manner that excludes negative numbers. It was unusual to find any mention of negative numbers within the subtraction sections of the books of this time period. However, there were some interesting exceptions. Cocker (1702) did mention the existence of negative numbers. Thomas Weston (1729) included “negative quantities and imaginary numbers” in his explanation of subtraction. These kinds of entries were rarely to be found, which makes Cocker’s and Weston’s presentations particularly interesting as they are from the late 1600s and early 1700s, the early part of our period of study.

The types of subtraction in both the arithmetic books and cyphering books from the 1700s to the early 1900s commonly came in two forms: simple subtraction and compound subtraction (e.g., Dilworth, 1810). Simple subtraction is the subtraction of whole numbers. In some cases, simple subtraction also included the subtraction of decimals (e.g., Colburn, 1855). A section on compound subtraction would either immediately follow the simple subtraction section in the books or would appear slightly later in the book. Compound subtraction, which was a direct application of simple subtraction, typically included problems about quantities relating to dry measures, land measures, money, and other important contexts of the time period. The subtraction was called “compound” because it was composed of a variety of units. For example, suppose that there are 12 pence in a shilling. Subtracting 1 shilling and 7 pence from 3 shillings and 5 pence is an example of a compound subtraction problem. Much rarer was Ayres’ (1711) definition of simple subtraction as subtracting single digit numbers and compound subtraction as subtracting compound numbers, which he defined as whole numbers composed of two, three, four or more digits. The research focus of this paper is on *simple subtraction* of whole numbers with any number of digits, although similar algorithms can be found in discussions of compound subtraction as well.

Minuend, Subtrahend, and Remainder

OPERATION.

From 237 *the minuend,*
Take 114 *the subtrahend,*

123 *the remainder.*

Figure 3. Minuend, subtrahend, and remainder in *Adam's* [sic] *New Arithmetic,* by Daniel Adams, published in Keene, New Hampshire, in 1830. (This image has been reproduced, with permission, from the arithmetic book belonging to Nerida F. Ellerton and M. A. (Ken) Clements.)

Language issues in subtraction and indeed in all of arithmetic deserve historical investigation, but will not be developed in this paper. We introduce only the terms the reader needs in order to understand the statements present in the texts and cyphering books. Authors of this time period utilized words such as "minuend," "subtrahend," and "remainder," largely absent from our modern curriculum. The minuend is the larger number and the one that will be "lessened." The word "minuend" comes from Latin "minuere" which means "to lessen" (Wingate, 1865). The subtrahend is the smaller number or the number being subtracted, and the remainder is the difference between the two numbers.

Although it was a very familiar sight to see the language of minuend, subtrahend, and remainder used in the 1700s to early 1900s, not everyone applied this terminology. Many books used "lesser number" and "greater number" or "lower figure" and "upper figure" rather than "minuend" and "subtrahend" (Brookes, 1776; Colburn, 1824; Daboll, 1829; Dilworth, 1802; Dilworth, 1810; Farrar, 1818; Lee, 1797; Pike, 1809). The words minuend and subtrahend did appear in early texts during this time period, and by the mid-1800s this terminology not only appeared, but was preferred over other terms. All of the texts examined after 1848 used the terms "minuend" and "subtrahend" (Adams, 1848; Colburn, 1855; Ray, 1856; Colburn, 1858; Wingate, 1865; Ray, 1877), but, again, these words are not often used in the modern classroom.

Algorithms Revealed

Four subtraction algorithms were used in different parts of Europe and America in the 18th and 19th centuries. They are the equal additions, decomposition, complement, and Austrian algorithms, and all four algorithms have been identified in the arithmetic books and cyphering books from this time period. Twentieth and 21st century historical research on subtraction algorithms has largely overlooked the complement algorithm and has not emphasized the scarcity of the Austrian algorithm in North America (Osburn, 1927; Johnson, 1938; Ross & Pratt-Cotter, 2000; Smith, 1909).

Mathematics educators of the past have recognized five or six subtraction algorithms. David Eugene Smith (1905), an influential historian of mathematics education, wrote in his *Handbook to Smith's Arithmetic*: "There are at least three or four methods in common use throughout the country" (p. 56). By 1909 he had increased his total to five algorithms, adding the "left-to-right method" to his earlier list of methods (Smith, 1909). Smith claimed, without development of the algorithm, that the left-to-right method "is more adapted to the needs of a professional computer, however, than to those of the average citizen, and may therefore be dismissed with this mere mention" (Smith, 1909, p. 77).
W. J. Osburn (1927) classified subtraction into three main categories: additive, take-away, and complementary. Osburn's "additive" approach is referred to in this paper as the Austrian algorithm. Surprisingly, Osburn grouped both the decomposition and equal additions algorithms into his "take away" category. Osburn's definition of the complementary method of subtraction is the same definition that is used in this paper. Within these general categories Osburn identified sub-methods based on the syntax of the subtraction problems. He considered "2 from 8" and "8 take away 2" as "upward" and "downward" subtraction, respectively, and constructed six different methods of subtraction based on this language. Based on the actual data from the arithmetic textbooks and cyphering books, I will discuss the four established algorithms:

- the equal additions algorithm,
- the decomposition algorithm,
- the complement (or complementary) algorithm, and
- the Austrian algorithm.

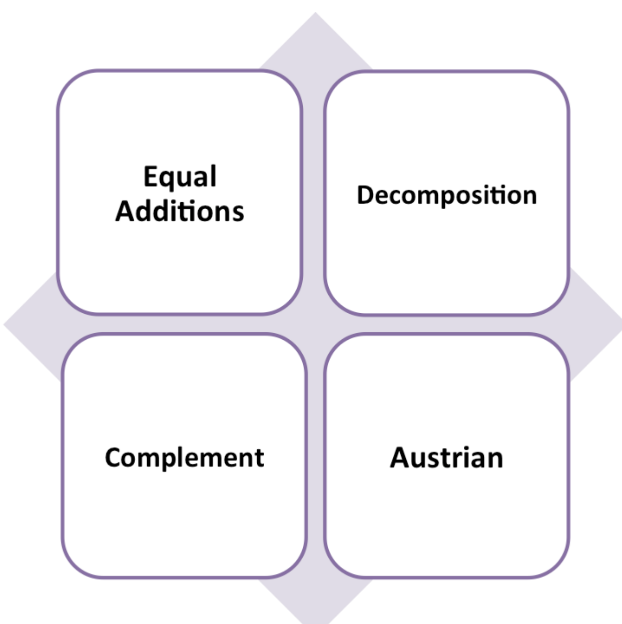


Figure 4. The four subtraction algorithms to be discussed in this article.

Example of the Equal Additions Algorithm

Consider the subtraction problem, $940-586$, or

	9	4	0
−	5	8	6

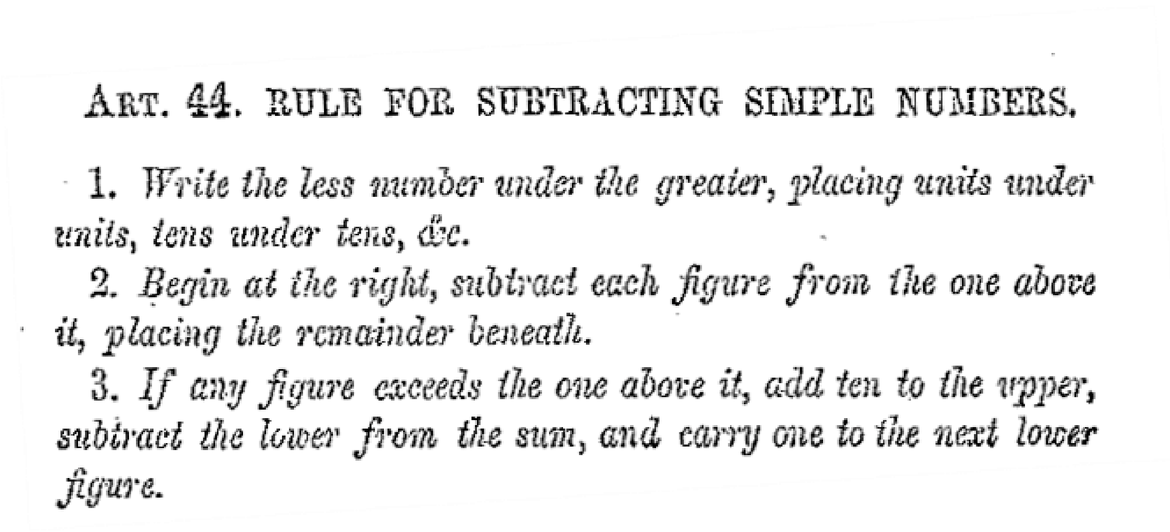
To begin subtraction with the Equal Additions Algorithm, ten would be added to the minuend in the ones position and ten would be added to the subtrahend by placing a one in the tens position. Using modern notation, we may add markings as shown in the video. In the ones position, instead of $0-6$, we now have $10-6$, giving us 4, since we added 10 to the ones position. In the tens position, instead of $4-8$, we now have $4-9$ since we added 1 in this position. Because we have $4-9$, we would need to apply the Equal Additions Algorithm again.

See the Equal Additions Algorithm in action using the example $940-586$:

Historical Discussion

The so-called “equal additions” algorithm was the most widely used algorithm from 1700 to 1900 (Ellerton & Clements, 2012). The algorithm obtained its name because the method essentially adds an equal number, ten, to both the minuend and the subtrahend. Thus, this algorithm is primarily based upon the identity $a-b=(a+k)-(b+k)$. David Eugene Smith (1909) called the equal additions method “the borrowing and repaying plan.” He stated that this algorithm has roots dating back to Pietro Borghi (also known as [Piero Borgi]), who, in 1484, wrote the first great business arithmetic ever printed. The equal additions algorithm was also used by arithmeticians in India and in Constantinople before printing was invented (Smith, 1909). Jackson (1906, cited in Johnson, 1938) stated that the equal additions algorithm could be found in many sixteenth century texts.

Joseph Ray (1856) provided a rule for “subtracting simple numbers” that is commonly found in nearly all texts supporting the equal additions algorithm (See Figure 5).


ART. 44. RULE FOR SUBTRACTING SIMPLE NUMBERS.

1. *Write the less number under the greater, placing units under units, tens under tens, &c.*

2. *Begin at the right, subtract each figure from the one above it, placing the remainder beneath.*

3. *If any figure exceeds the one above it, add ten to the upper, subtract the lower from the sum, and carry one to the next lower figure.*

Figure 5. Instructions for the equal additions algorithm on page 23 of *Ray’s higher arithmetic: The principles of arithmetic,* by Joseph Ray, published in Cincinnati, Ohio, in 1856. (This image has been reproduced from a Google Book with free access.)

Part three of the rule stated above is a typical statement. From 1700 to 1900, it was quite common to encounter the equal additions algorithm, as well as any other algorithm, with no explanation of why the algorithm worked (Ayres, 1711; Botham, 1835; Brookes, 1776; Cocker, 1677; Daboll, 1829; Dilworth, 1802; Greenwood, 1729; Ray, 1856/1877; Record, 1658; Walsh, 1828; Weston, 1729; Wingate, 1865).

Let it be required to Subtract 3872 *from* 43758.

The given Numbers being placed, and a Line drawn under them, as is before directed, I begin at the right hand, saying, 2 from 8, and there remains 6, which I set under the Line, and proceed, saying, 7 from 5 I cannot, but 7 from 15 and there remains 8, which I put under the Line, and proceed to the next, saying, 1 that I borrow'd and 8 is 9 from 7 I cannot, but 9 from 17 and there remains 8, which I put under the Line, and proceed to the next Figure, saying, 1 that I borrowed and 3 is 4, from 3 I cannot, but 4 from 13 and there remains 9, which I put under the Line; now because there is no Figure stand-

$$\begin{array}{r} 43758 \\ 3872 \\ \hline 39886 \end{array}$$

ſtanding under the 4, I therefore ſuppoſe a (0) Cypher to be placed there, and becauſe I borrowed 1 at the laſt Figure, therefore I pay it here by Subtracting it out of the 4, ſaying, 1 that I borrowed out of 4 and there remains 3, which I put under the Line, and the Work is finiſhed; and I find (after the Work of Subtraction is ended) the remainder to be 39886. Theſe Examples being well underſtood, will render what follows to be plain and eaſie.

From	7458	50876	10008576
Subtr.	467	947	8743
Remains	6991	49929	9999833
From	5100	30210	15764
Take	1754	10325	7276
Reſt	3346	19885	8488
Proof	5100	30210	15764

The Proof of Subtraction.

Figure 6. This example of the equal additions algorithm appears on pages 35 and 36 of *Arithmetick: A treatise defined for the use and benefit of trades-men* (11th edition), by John Ayres, published in London, England, in 1711. Ayres published the first edition of this text in 1693 as *Arithmetick: A treatise fitted for the use and benefit of such trades-men as are ignorant in that art*. (These images have been reproduced from a Google Book with free access.)

Example of the Decomposition Algorithm

Consider again the same subtraction problem as on the preceding page, $940-586$, or

	9	4	0
−	5	8	6

Nearly all students in the United States today utilize a method that begins with "borrowing" one group of ten from the 4 in the minuend, exchanging it for ten units, and giving these ten units to the 0 in the minuend. This is called the Decomposition Algorithm because, to subtract, students initially decompose the number 940 to $900+30+10$. One would need to borrow again from the 9 in the minuend. Thus, the decomposition is written as $(800+130+10)-(500+80+6)$.

See the Decomposition Algorithm in action using the example $940-586$:

Historical Discussion

The decomposition algorithm is the method of subtraction used predominantly today in the United States, and is referred to as the "standard subtraction algorithm" in many current texts. This algorithm is quite old, dating back to Spain in the thirteenth century, Italy in the Middle Ages, and India even earlier (Smith, 1909). In fact, there is evidence of the decomposition algorithm present in the writings of Rabbi ben Ezra (1140, cited in Smith, 1925). Prior to being called decomposition, this algorithm was also referred to as "simple borrowing" (Smith, 1909). The "borrowing" in this algorithm is different than the "borrowing" utilized in equal additions. In the equal additions algorithm, ten is added to the minuend and then "repaid" by adding it to the subtrahend. On the contrary, with the decomposition algorithm no numbers are added to the subtrahend or minuend; rather, the subtrahend and minuend are simply "decomposed" and the decomposition of the minuend re-arranged. For example, $41-29$ can be thought of as $(30+11)-(20+9)$. Then, the units are subtracted and the tens are subtracted, as follows:

$$(30+11)-(20+9)=(30-20)+(11-9)=12.$$

Chauncey Lee, a popular author in the 1790s, advocated the decomposition algorithm, writing (Lee, 1797, p. xi):

This (decomposition method) I conceive to be a more simple, natural and easy mode, especially in whole numbers.

Warren Colburn (1824) also encouraged the use of the decomposition algorithm. Colburn's explanation, in Figure 7, is distinct from all the other texts because he actually demonstrated the "decomposing."

A man having 62 sheep in his flock, sold 17 of them; how many had he then?

Operation.

He had 62 sheep
Sold 17 sheep
—
Had left 45 sheep

In this example a difficulty immediately presents itself, if we attempt to perform the operation as before; for we cannot take 7 from 2. We can, however, take 7 from 62, and there remains 55; and 10 from 55, and there remains 45, which is the answer.

The same operation may be performed in another way, which is generally more convenient. I first observe, that 62 is the same as 50 and 12; and 17 is the same as 10 and 7. They may be written thus:

$$62 = 50 + 12$$
$$17 = 10 + 7$$
$$45 = 40 + 5$$

That is, I take one ten from the six tens, and write it with the two units. But the 17 I separate simply into units and tens as they stand. Now I can take 7 from 12, and there remains 5. Then 10 from 50, and there remains 40, and these put together make 45.*

* Let the pupil perform a large number of examples by separating them this way, when he first commences subtraction.

13*

Figure 7. Decomposition algorithm on p. 150 of *First lessons in arithmetic* (2nd ed., 1824), by Warren Colburn. (This image has been reproduced from a Google Book with free access.)

Probably influenced by Colburn, Daniel Adams (1830) also supported the decomposition algorithm. Adams' explanation, shown in Figure 8, was typical of the time period, with no markings and no decomposition illustrated. Also of special interest in the passage in Figure 8 is that Adams presented the equal additions algorithm at the end of this excerpt as an alternate method.

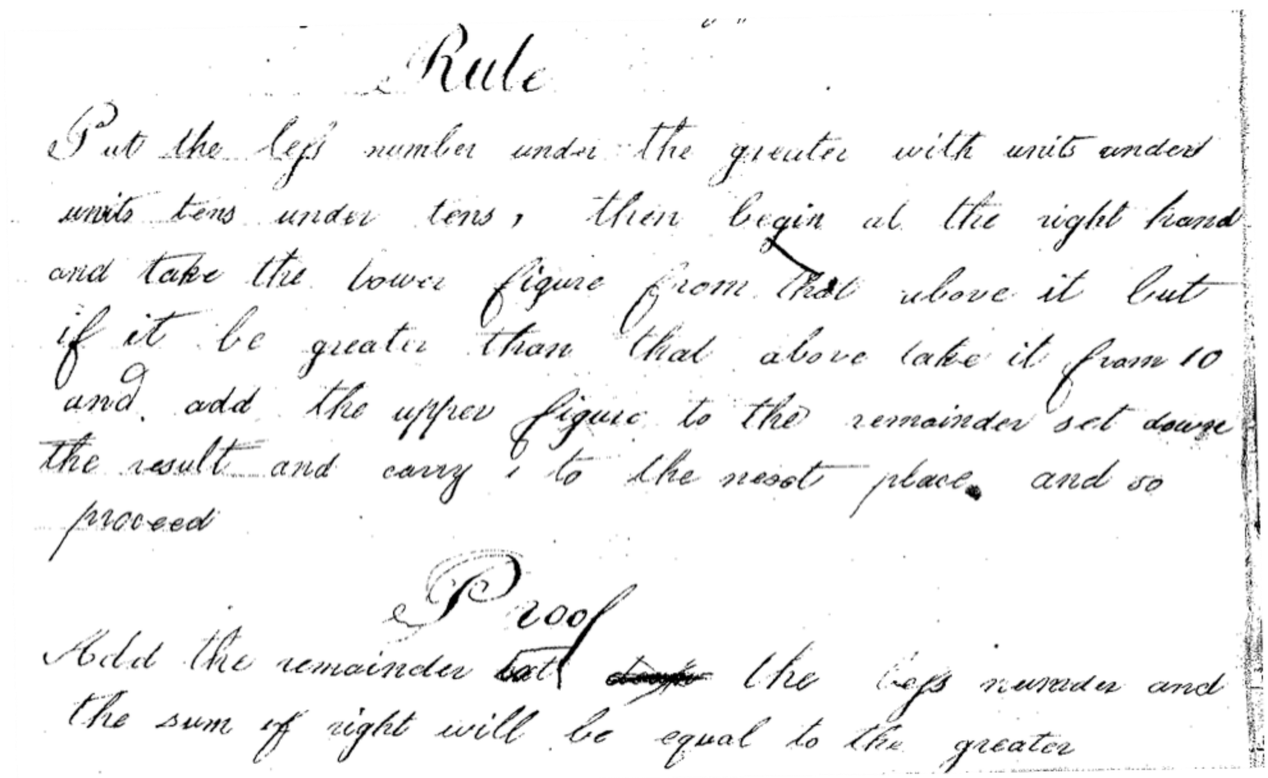
18. A man bought a horse for 85 dollars, and a cow for 27 dollars; what did the horse cost him more than the cow?

OPERATION.

Horse,	85
Cow,	27
Difference,	58

The same difficulty meets us here as in the last example; we cannot take 7 from 5; but in the last example the larger number consisted of 1 ten and 5 units, which together make 15; we therefore took 7 from 15. Here we have 8 tens and 5 units. We can now, in the mind, suppose 1 ten taken from the 8 tens, which would leave 7 tens, and this 1 ten we can suppose joined to the 5 units, making 15. We can now take 7 from 15, as before, and there will remain 8, which we set down. The taking of 1 ten out of 8 tens, and joining it with the 5 units, is called *borrowing ten.* Proceeding to the next higher order, or tens, we must consider the upper figure, 8, from which we borrowed, 1 less, calling it 7; then, taking 2 (tens) from 7, (tens,) there will remain 5, (tens,) which we set down, making the difference 58 dollars. Or, instead of making the *upper* figure 1 *less*, calling it 7, we may make the *lower* figure one *more*, calling it 3, and the result will be the same; for 3 from 8 leaves 5, the same as 2 from 7.

Figure 8. Typical explanation of decomposition algorithm, with the equal additions algorithm suggested at the end as an alternate method, on p. 22 of *Adam's* [sic] *New Arithmetic,* by Daniel Adams, published in Keene, New Hampshire, in 1830. (This image has been reproduced, with permission, from the book belonging to Nerida F. Ellerton and M. A. (Ken) Clements.)



Rule

Put the less number under the greater with units under units tens under tens, then begin at the right hand and take the lower figure from that above it but if it be greater than that above take it from 10 and add the upper figure to the remainder set down the result and carry 1 to the next place and so proceed

Proof

Add the remainder ~~bat~~ ~~[illegible]~~ the less number and the sum of right will be equal to the greater

Figure 9. Complement algorithm in the cyphering book of Jonathan Butlar, written in 1785 in Bucks County, Pennsylvania. (This image is reproduced, with permission, from the cyphering book belonging to Nerida F. Ellerton and M. A. (Ken) Clements.)

Note on source: For details about this and many other cyphering books, consult *Rewriting the History of School Mathematics in North America 1607–1861,* by N. F. Ellerton and M. A. Clements (Springer, 2012). Ellerton and Clements have indicated that photographs of excerpts from the cyphering books in their collection can be reproduced in scholarly papers or presentations or in curriculum materials, provided it is indicated that the cyphering books from which the photographs were taken are in the Ellerton-Clements collection and provided reference is made to the above-mentioned book by Ellerton and Clements (2012).

Although the complement algorithm appeared as early as 1478 in the *Treviso Arithmetic* (Smith, 1909; Swetz, 1987), 20th and 21st century mathematics educators, at least during the last 100 years or so, have considered the algorithm to be a special case of the equal additions algorithm or have forgotten it entirely. It seems that by the 1930s the "complementary method" was losing popularity. J. T. Johnson (1938) performed a study comparing subtraction algorithms and omitted the complement algorithm from the study. More recently, Susan Ross and Mary Pratt-Cotter (2000) provided an example of the equal additions algorithm in their article, "Subtraction in the United States: A Historical Perspective," which is actually an example of the complement method. Willetts (1819, p. 11, as cited in Ross & Pratt-Cotter, 2000) gave the following rule for "subtraction":

1. Begin at the right hand and take the lower figure from the one above it and set the difference down.
2. Place the lesser number under the greater, with the units under the units, tens under tens, etc.
3. If the figure in the lower line be greater than the one above it, take the lower figure from 10 and add the difference to the upper figure which sum set down.
4. When the lower figure is taken from 10 there must be one added to the next lower figure.

Ross and Pratt-Cotter classified this rule as demonstrating the equal additions algorithm. Upon closer examination of this algorithm, one can see that it is different from the equal additions algorithm in the third step. When executing the subtraction, one is calculating the complement using the digit in the subtrahend and then adding the digit in the minuend. Algebraically, with the complement method, if one is given

$$(10a+b)-(10c+d), d>b,$$

then

$$(10a+b)-(10c+d)=((10-d)+b)+10(a-(c+1)).$$

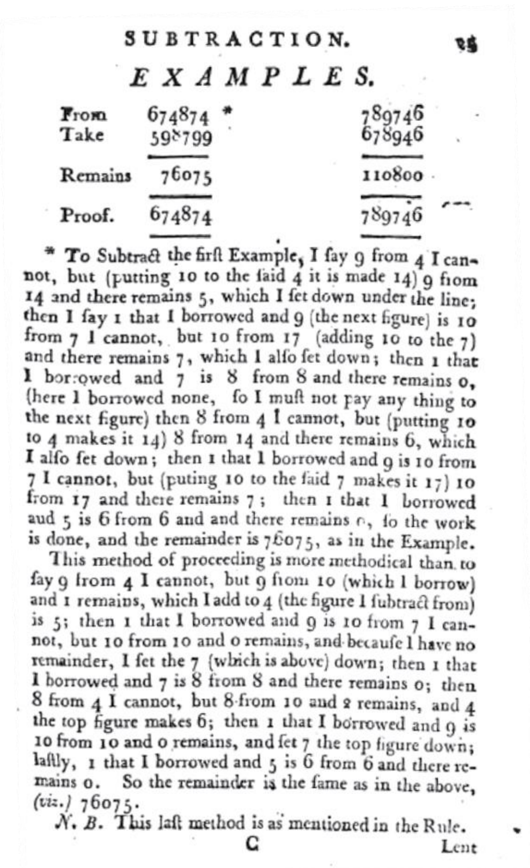
Whereas, with the equal additions algorithm, if one is given

$$(10a+b)-(10c+d), d>b,$$

then

$$(10a+b)-(10c+d)=(b+10-d)+10(a-(c+1)).$$

To be more concise, the complementary method of subtraction is based on the identity $a-b=a+(10-b)-10$(Smith, 1925), whereas the equal additions algorithm is based on the identity $a-b=((a+10)-b)-10$.Admittedly, the algorithms are closely related, especially to the modern mathematician's eye; however, mathematicians and arithmeticians from the 1700s and 1800s viewed the complementary method as a different algorithm from the equal additions algorithm. J. Brookes (1776) mentioned both the equal additions algorithm and the complementary algorithm in his book. He clearly identified both algorithms as different methods of subtraction and clearly stated a preference for the complementary method. Brookes (1776) referred to the equal additions algorithm as a "more methodical" algorithm (p. 25).



SUBTRACTION. 25

EXAMPLES.

From	674874 *	789746
Take	598799	678946
Remains	76075	110800
Proof.	674874	789746

* To Subtract the first Example, I say 9 from 4 I cannot, but (putting 10 to the said 4 it is made 14) 9 from 14 and there remains 5, which I set down under the line; then I say 1 that I borrowed and 9 (the next figure) is 10 from 7 I cannot, but 10 from 17 (adding 10 to the 7) and there remains 7, which I also set down; then 1 that I borrowed and 7 is 8 from 8 and there remains 0, (here I borrowed none, so I must not pay any thing to the next figure) then 8 from 4 I cannot, but (putting 10 to 4 makes it 14) 8 from 14 and there remains 6, which I also set down; then 1 that I borrowed and 9 is 10 from 7 I cannot, but (puting 10 to the said 7 makes it 17) 10 from 17 and there remains 7; then 1 that I borrowed aud 5 is 6 from 6 and and there remains 0, so the work is done, and the remainder is 76075, as in the Example.

This method of proceeding is more methodical than to say 9 from 4 I cannot, but 9 from 10 (which I borrow) and 1 remains, which I add to 4 (the figure I subtract from) is 5; then 1 that I borrowed and 9 is 10 from 7 I cannot, but 10 from 10 and 0 remains, and because I have no remainder, I set the 7 (which is above) down; then 1 that I borrowed and 7 is 8 from 8 and there remains 0; then 8 from 4 I cannot, but 8 from 10 and 2 remains, and 4 the top figure makes 6; then 1 that I borrowed and 9 is 10 from 10 and 0 remains, and set 7 the top figure down; lastly, 1 that I borrowed and 5 is 6 from 6 and there remains 0. So the remainder is the same as in the above, (*viz.*) 76075.

N. B. This last method is as mentioned in the Rule.

C Lent

Figure 10. Author J. Brookes preferred the complement algorithm, as indicated on page 25 of his *A treatise on arithmetic*, published in 1776 in New Castle, England. (The image of this page has been reproduced from a Google Book with free access.)

Example of the Complement Algorithm

Let's consider subtracting 19 from 26, or

	2	6
−	1	9

using the Complement (or Complementary) Algorithm. To perform the subtraction $26-19$ using this algorithm, one gives 10 to the minuend and 10 to the subtrahend, as in the Equal Additions Algorithm. Where the Complement Algorithm differs from the Equal Additions Algorithm is, not surprisingly, in its use of the complement with respect to 10.Instead of computing $16-9=7$, one computes $(10-9)+6=7$, where $10-9$ is the complement with respect to 10. Then, one computes $2-2=0$.

See the Complement Algorithm in action using the example $940-586$:

Historical Discussion

The complement, or complementary, algorithm was another widely used algorithm during the 1700s and 1800s. Thomas Dilworth, a well-known author whose arithmetic text went through several editions, advocated the use of this algorithm, writing (Dilworth, 1802, p. 21),

When the lower number is greater than the upper, take the lower number from the number which you borrow, and to that difference add the upper number, carrying one to the next lower place.

As shown in Figure 9, Jonathan Butlar (1785) presented the complement algorithm in his cyphering book as follows:

Put the less number under the greater with units under units tens under tens, then begin at the right hand and take the lower figure from that above it but if it be greater than that above take it from 10 and add the upper figure to the remainder set down the result and carry 1 to the next place and so proceed.

Example of the Austrian Algorithm

Consider the subtraction problem, $56-19$, or

	5	6
−	1	9

The Austrian Algorithm for subtraction is quite intuitive, taking advantage of subtraction as the inverse of addition. In this example, one would first consider what whole number should be added to 9 to obtain 6. Since this is not possible with only whole numbers, one would then consider the whole number that could be added to 9 to obtain 16, which is 7. To compensate for the 10, one would now look at the 1 as a 2. Using similar logic, 2 plus a whole number would need to give 5, which gives 3 in the tens place and a difference of 37.

See the Austrian Algorithm in action using the example $940-586$:

Historical Discussion

Due to efforts to introduce this algorithm into Austrian schools and subsequently German schools, the algorithm acquired the name of "Austrian method." The algorithm has also been called the "addition" or "making change" method (Smith, 1909) because it utilizes addition. One must think of what must be added to the subtrahend to get the minuend. This algorithm is again illustrated in the following excerpt (McClellan & Ames, 1902, as cited in Johnson, 1938, p. 23):

From 94,275 take 67,492:

94275
67492
26783

Thus: 2 and 3 are 5; 9 and 8 are 17, carry 1 to 4 as in addition, making it 5, 5 and 7 are 12; carry 1 to 7 making it 8; 8 and 6 are 14; carry 1 to 6 making it 7; 7 and 2 are 9.

Only one German text from 1700 to 1900 in the Ellerton-Clements collection of arithmetic books used the Austrian method of subtraction, that of Albert Braune, published in Leipzig, Germany, in 1882. This book, written in German, may have been utilized by German-speakers in the United States. J. T. Johnson (1938) wrote that the excerpt given above was the earliest use of the Austrian method in the United States, but the existence of the Braune book may disprove this claim. Johnson also wrote (1938, p. 23),

The strictly addition procedure in subtraction is mentioned in the Handbuch der Mathematik by [Adam] Bittner, published in Prague in 1821; this method is explained in [Joseph] Salomon's (1849) Lehrbuch der Arithmetik und Algebra.

Based purely on inference from Johnson's research and the books examined in this research, the Austrian method, while possibly prevalent in books in Germany and Austria and potentially within some books in the United States, was not often used as an algorithm in the United States between 1700 and 1900.

The equal additions algorithm, decomposition algorithm, complement algorithm, and Austrian algorithm for subtraction are represented in printed arithmetic books between 1700 and 1900. Was one algorithm preferred over the other algorithms during this time period? Table 1 shows our sample of printed arithmetic books examined and lists the algorithms that were utilized in them. If a book had more than one algorithm in it, it was listed more than once and has an asterisk next to its name.

Table 1. ***Summary of Algorithms Used in Arithmetic Books from 1700 to 1900***

Equal Additions	**Decomposition**	**Complement**	**Austrian**
Record (1658)[‡]	Weston (1729)*	Brookes (1776)*	Braune (1882)
Cocker (1702)	Barreme (1747)	Dilworth (1802/1810)	
Ayres (1711)	Lee (1797)*	Gough (1803)	
Weston (1729)*	Farrar (1818)	Pike (1809)	
Brookes (1776)*	W. Colburn (1824)	Lee (1797)*	
Walsh (1828)	Adams (1830/1845)		
Daboll (1829)	Emerson (1832)		
Botham (1835)	D. P. Colburn (1855/1858)		
Ray (1856/1877)*	Ray (1856/1877)*		
Wingate (1865)	Fish (1874)*		
Fish (1874)*	Wentworth (1897)		

Thomas Weston (1729) mentioned both the equal additions algorithm and the decomposition algorithm in his text without indicating a preference for either algorithm. Equal additions and decomposition are present in Joseph Ray's (1856/1877) texts. Ray (1856) included both algorithms, presenting the equal additions algorithm first and the decomposition algorithm second. Later, Ray (1877), despite introducing the decomposition algorithm first, gave preference to the equal additions algorithm (see Figure 12). Figure 12 shows Ray's explanation of how to compute $805-637$ using his second method, equal additions.

2d Method.—If the 5 units be increased by 10, say 7 from 15 leaves 8; then, increasing the 3 by 1, say 4 from 0 can not be taken, but 4 from 10 leaves 6; then, increasing 6 by 1, say 7 from 8 leaves 1.

REM. 1.—The second method is generally used; it is more convenient, and less liable to error, especially when the upper number contains ciphers.

Figure 11. Subtraction using equal additions, with preference given to this method “especially when the upper number contains ciphers [zeros],” on p. 35 of *Ray’s new practical arithmetic,* by Joseph Ray, published in Cincinnati, Ohio, in 1877. (This image has been reproduced from a Google Book with free access.)

Ray may have provided two explanations but he definitely advocated for equal additions by stating the equal additions algorithm was “less liable to error.” Ray was not alone in this opinion (see, e.g., Osburn, 1928). There were numerous studies in the early 1900s comparing algorithms and many believed the equal additions algorithm to be the “preferred” method. While many claimed that the equal additions algorithm was “less liable to error,” not everyone agreed that this was the case. Chauncey Lee (1797) called the equal additions algorithm “circuitous” and stated that the decomposition algorithm was a “more simple, natural and easy mode” (p. xi). J. Brookes (1776) preferred the complement method. J. T. Johnson (1938, p. 27) wrote:

[T]he equal additions method was not found in any German text examined, the decomposition method being chiefly used in that country. On the other hand, the decomposition method was not found in any French book examined, published later than 1820, the equal additions method being the prevalent method in France. The complementary and equal additions were the outstanding methods in Italy and England. The Austrian method was found in very recent English texts but did not appear in any Italian books examined.

It is interesting to note that all four algorithms – equal additions, complement, decomposition, and Austrian – are represented in printed books probably used in the U.S., as shown in Table 1.

Despite the presence of all four algorithms in printed books with authors as advocates for each, only two algorithms are present in cyphering books. When Nerida Ellerton and Ken Clements (2012) analyzed their private collection of 280 cyphering books, they found 51 of the cyphering books specifically stated a rule for “subtraction.” Of those 51 cases, 33 described “equal additions” and 18 the “complementary method.” The fact that not a single cyphering book in Ellerton’s and Clements’ personal collection of 280 cyphering books, dated between 1701 and 1860, utilized the decomposition algorithm indicates that decomposition was perhaps not an implemented subtraction algorithm in North America during this time. If it can be assumed that cyphering books displayed what students actually studied at the time, the fact that not a single cyphering book contains the decomposition algorithm prompts two very important questions. First, why was the decomposition algorithm not widely used from the 1700s to the early 1900s in cyphering books despite prominent authors, such as Warren Colburn, advocating the algorithm? And, second, given that the decomposition algorithm is prevalently used and advocated in modern classrooms, when did this transition occur?

In the early 1900s there was not a “standard” subtraction algorithm as there is today. In fact, in his *Handbook to Smith’s Arithmetic,* David Eugene Smith recommended no specific algorithms. According to Smith (1905), “for a book to insist upon one of these would be to confuse the children in a school where another method is working satisfactorily” (p. 56). Despite Smith not recommending a “standard” method, there was a “great debate” in the early 1900s upon what should be the chosen algorithm (Johnson, 1938; Osburn, 1927; Ross & Pratt-Cotter, 2000). Many studies were devoted to determining which algorithm caused the least error and in the early 1900s the majority view was that the equal additions algorithm was superior (Johnson, 1938; Osburn, 1927). In fact, W. J. Osburn asserted at the end of his study (1927, p. 246):

The superiority of equal-additions (carrying) over decomposition (borrowing) is 'as certain as taxes' and 'almost as certain as death'.

With the equal additions algorithm prominent in cyphering books and with studies in the early 1900s supporting the superiority of the equal additions algorithm, it is quite surprising that this algorithm is hardly present today in schools in the United States.

The ideas of John Heinrich Pestalozzi, a Swiss-German educator, were a possible influence for the selection of the decomposition algorithm. During the period from 1820 to 1880 Pestalozzi's ideas became well known and around this same time the algorithms debate began (Ellerton & Clements, 2012). Osburn (1927) argued that one of the reasons for the increasing popularity of decomposition was directly tied to Pestalozzi (Osburn, 1927, p. 242):

It was the Pestalozzian movement for object lesson which brought in the decomposition method, the reason being that decomposition can be illustrated with bundles of splints.

Both Pestalozzi and Colburn urged other teachers to adopt an inductive approach, where the rules or algorithms did not precede the learning, an approach the current cyphering books and printed books did not support (Ellerton & Clements, 2012). The influences of normal-school reformers, the introduction of written examinations into schools for children, and in particular the implementation of the first "standardized" examinations by Horace Mann are all possible reasons not only for the changes in terminology used, but also for the need to use a "standardized" algorithm. The definitive reasons for switching from the equal additions algorithm to the decomposition algorithm are not clear and deserve further study.

‡Although Robert Record, or Recorde, first published his *Grounde of Artes* in London in 1542, the 1658 copy was examined. During this time, additions and changes to books often were included as additional pages at the end of the book. The editing process for Record's *Grounde of Artes* seems to have worked this way, keeping the original intact and just adding a new section to the back. Record died in 1558; therefore, 17th century publications of his arithmetic book were most likely revised by others. Images from the 1543 edition of the *Grounde of Artes* can be seen here in *Convergence*.

Implications for the Modern Teacher

The teaching and learning of subtraction is just as important today as it was in the past. Innovations in technology and mathematics curriculum have certainly occurred since the 1700s and 1800s, but the need for the teaching and learning of subtraction has not changed. Today, in many classrooms, subtraction is often taught through student-invented algorithms. Looking to the past may give teachers insight into invented algorithms or other algorithms students may use. Additionally, many teachers who do not encourage students to invent strategies teach only the "standard subtraction algorithm" presented in nearly every textbook across the United States, the decomposition algorithm. This research and analysis provides the modern teacher with an opportunity to reflect on the algorithms being taught in his or her classroom and allows the teacher to begin to think about why decomposition became the dominant algorithm in the United States. Teachers can ask their students to reflect on whether they agree with this historical turn of events. Incorporating the history of subtraction algorithms into modern elementary school mathematics invites robust mathematical discussion of subtraction and also of how, for many mathematical operations, there isn't just one algorithm, but rather many algorithms from which to choose.

Exploring the history of subtraction in past school mathematics may provide us with insight into students' mathematical struggles as they attempt to conceptualize not only subtraction, but also negative numbers and other notoriously challenging mathematical concepts. As educators and researchers, we need to devote more attention to issues in mathematics education such as the development of specific algorithms in elementary mathematics.

Acknowledgments

The images of pages of cyphering books in this article have been reproduced, with permission, from cyphering books belonging to Nerida F. Ellerton and M. A. (Ken) Clements, who, as of January 2014, own a collection of 350 North American cyphering books, dating from 1701 to 1861. This is the largest collection, public or private, of North American cyphering books. For details relating to any of these cyphering books, consult *Rewriting the History of School Mathematics in North America 1607–1861,* by N. F. Ellerton and M. A. Clements (Springer, 2012). Ellerton and Clements have indicated that photographs of excerpts from the cyphering books in their collection can be reproduced in scholarly papers or presentations or in curriculum materials, provided it is indicated that the cyphering books from which the photographs were taken are in the Ellerton-Clements collection and provided reference is made to the above-mentioned book by Ellerton and Clements (2012).
Nerida F. Ellerton and M. A. (Ken) Clements also own a collection of arithmetic textbooks that I have consulted and from which I have used images. I would also like to thank Dr. Ellerton and Dr. Clements not only for the use of their arithmetic and cyphering books, but also for their feedback on this project. They have a passion for the history of mathematics education and have been an inspiration to me.

About the Author

Nicole Wessman-Enzinger is a Ph.D. candidate in mathematics education within the Department of Mathematics at Illinois State University. She is working on her dissertation about developing conceptual models of student thinking about negative integers. Her dissertation is focused on fifth graders' thinking about negative integers and operations with them. She is broadly interested in the teaching and learning of number, from both historical and psychological perspectives.

Arithmetic Books (1658 – 1882)

Adams, D. (1808). *The scholar's arithmetic or federal accountant* (5th ed.). Troy, NY: John Prentiss.
Adams, D. (1830). *Adam's new arithmetic*. Keene, NH: J. & J.W. Prentiss.
Adams, D. (1848). *Adam's new arithmetic: Revised edition*. Keene, NH: J. & J.W. Prentiss.
Ayres, J. (1711). *Arithmetick: A treatise defined for the use and benefit of trades-men* (11th ed.). London, UK: Tho. Norris at Looking Glass on London Bridge.
Barreme, N. (1747). *L'Arithmetique du Sr Barreme ou le livre facile pour apprendre l'arithmetique.* Paris, France: Gandouin.
Botham, P.E. (1835). *Botham's common school arithmetic: The common school arithmetic.* Hartford, CT: Henry Benton.
Braune, A. (1882). *Vollständige Kaufmännische Arithmetik für Handels-, Real- und Gewerbescheulen sowie zum Selbstunterrichte für Angehende Geschäftsleute.* Leipzig: Verlag von Ferdinand Hirt & Sohn.
Brookes, J. (1776). *A treatise on arithmetic.* New Castle, UK: J. Smith.
Cocker, E. (1702). *Cocker's arithmetick*. London, UK: W. Richardson.
Colburn, W. (1824). *First lessons in arithmetic* (2nd ed.). Boston, MA: Hilliard & Metcalf.
Colburn, D. P. (1855). *Arithmetic and its applications: Designed as a text book for common schools, high schools, and academies.* Philadelphia, PA: H. Cowperthwait & Co.
Colburn, D .P. (1858). *The common-school arithmetic: A practical treatise on the science of numbers.* Philadelphia, PA: H. Cowperthwait & Co.
Daboll, N. (1829). *Daboll's schoolmaster's assistant, a plain practical system of arithmetic*. Utica, NY: Hastings & Tracy.
Dilworth, T. (1802). *The schoolmaster's assistant: Being a compendium of arithmetic, both practical and theoretical.* New York, NY: Lewis Nichols.
Dilworth, T. (1810). *The schoolmaster's assistant: Being a compendium of arithmetic, both practical and theoretical*. New York, NY: E. Duyckinck & R. McDermott.
Farrar, J. (1818). *Arithmetic taken primarily from the arithmetic of S.F. Lacroix*. Cambridge, MA: Hilliard & Metcalf.
Fish, D.W. (1874). *Robinson's shorter course: The complete arithmetic oral and written.* New York, NY: Ivison, Blakeman, & Co.
Fisher, G. (1785). *The instructor*. Edinburgh, Scotland: J. Dickson.
Gough, J. (1803). *Practical arithmetick*. Dublin, Ireland: Author.
Greenleaf, B. (1866). *The national arithmetic, on the inductive system, combining the analytic and synthetic methods; forming a complete course of higher arithmetic*. Boston, MA: Robert S. Davis & Co.
Greenwood, I. (1729). *Arithmetick vulgar and decimal with the application thereof to a variety of cases in trade and commerce.* Boston, MA: Kneeland & Green.
Lee, C. (1797). *The American accomptant: Being a plain, practical and systematic compendium of federal arithmetic.*Lansingburgh, NY: William W. Wands.
Pike, N. (1788). *The new complete system of arithmetic, composed for the use of the citizens of the United States.*Newburyport, MA: John Mycall.
Pinnock, William (1823). *Pinnock's catechism of arithmetic containing the most useful rules with exercises upon each* (6th ed.). London, UK: G. & W.B. Whittaker.
Ray, J. (1856). *Ray's higher arithmetic: The principles of arithmetic.* Cincinnati, OH: Winthrop B. Smith & Co.
Ray, J. (1877). *Eclectic educational series: Ray's new practical arithmetic.* Cincinnati, OH: Van Antwerp, Bragg & Co.
Record[e], R. (1658). *Record's arithmetick or the ground of arts.* London, England: James Mether.
[Spence, David] (1721). *Arithmetick compendiz'd or, a short treatise of arithmetick.* Edinburgh, Scotland: William Monro.
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Wentworth, G.A. (1897). *A practical arithmetic*. Boston, MA: Ginn & Company.
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Cuming, M. (c. 1820). RI: Unpublished manuscript.

Green, W. (1795-1800). Connecticut: Unpublished manuscript.

Hall, E. (1780). Unpublished manuscript.

Johnson, L. (1798-1801). MA: Unpublished manuscript.

Prust, T. (1702). Unpublished manuscript.

Steiner, M. (1801). Germany: Unpublished manuscript.

Senter, W. (1795-1801). Portsmouth, NH: Unpublished manuscript.

Thompson, J. (1820-1834). ME: Unpublished manuscript.

Townsend, W. (1824-1830). NY: Unpublished manuscript.

Unknown author (c. 1820). RI: Unpublished manuscript.

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Adams, D. (1848). *Adam's new arithmetic: Revised edition*. Keene, NH: J. & J.W. Prentiss.
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Braune, A. (1882). *Vollständige Kaufmännische Arithmetik für Handels-, Real- und Gewerbescheulen sowie zum Selbstunterrichte für Angehende Geschäftsleute.* Leipzig: Verlag von Ferdinand Hirt & Sohn.
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Cocker, E. (1702). *Cocker's arithmetick*. London, UK: W. Richardson.
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Dilworth, T. (1810). *The schoolmaster's assistant: Being a compendium of arithmetic, both practical and theoretical*. New York, NY: E. Duyckinck & R. McDermott.
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Green, W. (1795-1800). Connecticut: Unpublished manuscript.

Hall, E. (1780). Unpublished manuscript.

Johnson, L. (1798-1801). MA: Unpublished manuscript.

Prust, T. (1702). Unpublished manuscript.

Steiner, M. (1801). Germany: Unpublished manuscript.

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Adams, D. (1848). *Adam's new arithmetic: Revised edition*. Keene, NH: J. & J.W. Prentiss.

Ayres, J. (1711). *Arithmetick: A treatise defined for the use and benefit of trades-men* (11th ed.). London, UK: Tho. Norris at Looking Glass on London Bridge.

Barreme, N. (1747). *L'Arithmetique du Sr Barreme ou le livre facile pour apprendre l'arithmetique.* Paris, France: Gandouin.

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Braune, A. (1882). *Vollständige Kaufmännische Arithmetik für Handels-, Real- und Gewerbescheulen sowie zum Selbstunterrichte für Angehende Geschäftsleute.* Leipzig: Verlag von Ferdinand Hirt & Sohn.

Brookes, J. (1776). *A treatise on arithmetic.* New Castle, UK: J. Smith.

Cocker, E. (1702). *Cocker's arithmetick*. London, UK: W. Richardson.

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Thompson, J. (1820-1834). ME: Unpublished manuscript.

Townsend, W. (1824-1830). NY: Unpublished manuscript.

Unknown author (c. 1820). RI: Unpublished manuscript.

Wilson, U. (1823-1827). PA: Unpublished manuscript.

References (1905 – 2012)

Ellerton, N. F. & Clements, M. A. (2012). *Rewriting the history of school mathematics in North America 1607-1861*. New York, NY: Springer.

Johnson, J. T. (1938). *The relative merits of three methods of subtraction: An experimental comparison of the decomposition method of subtraction with the equal additions and the Austrian method.* New York, NY: Teachers College, Columbia University.

National Governors Association Center for Best Practices. Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington DC: Author. Retrieved from http://www.corestandards.org/the-standards

Osburn, W. J. (1927). How shall we subtract? *Journal of Educational Research,* 16(4): 237 -246.
Ross, S., & Pratt-Cotter, M. (2000). Subtraction in the United States: An historical perspective. *Mathematics Educator,*8(1), 49-56.

Smith, D. E. (1905). *Handbook to Smith's arithmetic.* Boston, MA: Ginn & Co.

Smith, D. E. (1925). *History of mathematics: Special topics of elementary mathematics* (Vol. 2). Boston, MA: Ginn & Co.

Smith, D. E. (1909). *The teaching of arithmetic.* Boston, MA: Ginn & Co.

Swetz, F. J. (1987). *Capitalism and arithmetic*. La Salle, IL: Open Court.

Nicole M. Wessman-Enzinger (Illinois State University) , "An Investigation of Subtraction Algorithms from the 18th and 19th Centuries - References," *Convergence* (January 2014)