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Singular Points of Real Sextic Curves I

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David A. Weinberg · Nicholas J. Willis

Abstract A complete classification of the individual types of singular points is given for irreducible real sextic curves. This classification is derived by using the computer algebra system Maple. There are 191 types of singular points for real irreducible sextic curves. We clarify that the classification is based on computing just enough of the Puiseux expansion to separate the branches. A significant portion of the proof consists of a sequence of large symbolic computations that can be done nicely using Maple.

Keywords Sextic curve · Singular point · Newton polygon · Puiseux expansion

Mathematics Subject Classification (2000) Primary 14H20 · Secondary 14B05 · 14P05

1 INTRODUCTION

The classification of simple singularities of irreducible complex sextic curves is due to Urabe and Yang (see [6, 7, 10]). They not only classified the individual types of simple singular points, but also all sets of simple singular points. They used Nikulin's theory of lattice embeddings and the theory of K3 surfaces. We will present the classification of all individual types of singular points for real (as well as complex) irreducible sextic curves. There are 191 types of singular points for irreducible real sextic curves and 108 types for irreducible complex sextic curves. Our derivation gives a very nice illustration of the role that computer algebra can play in doing proofs.

The general question is how shall we classify singular points of real sextic curves. For each fixed degree, we want a finite classification of singular points for all algebraic curves

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of that degree. Thus, in general, the local diffeomorphism type is not the desired criterion of classification. For example, in the Arnol'd notation, four lines intersecting at the origin represents an X_9 singular point. Up to local diffeomorphism, there are infinitely many equivalence classes in this family. We would like to regard the X_9 singular point as one type, and with respect to our criterion, it is one type. Notice here that an irreducible real quintic curve can have an X_9 singular point; thus, there are infinitely many local diffeomorphism types for irreducible real quintic curves.

Now let us describe how we will classify the individual types of singular points that a real algebraic curve can have. Given any polynomial equation $f(x, y) = 0$, it is possible to solve for y in terms of x in the form of fractional power series, called Puiseux expansions. There is an algorithm for doing this, and the software Maple computes such Puiseux expansions, even for curves with literal coefficients. Our classification will be based on taking just enough of the Puiseux expansions to separate the "branches," and noting the exponents at which the "branches" separate. In other words, compute the Puiseux expansions to a power of x such that all expansions are unique. Then we will associate a tree diagram, which will be described in detail below, that will codify how the "branches" separate and that will serve to classify the type of the singular point. It follows from Sect. 10 of Milnor's book [5] that such a classification gives a finite number of types for each fixed degree.

In studying a singular point of an algebraic curve, the first thing to look at is the Newton polygon. (Our Newton polygons will follow the style of Walker [8].) Corresponding to each segment of the Newton polygon, there is a quasihomogeneous polynomial [3, p. 195]. If all such quasihomogeneous polynomials have no multiple factors, then the Newton polygon already tells us the type of the singularity. (Note that in this case, we know right away the exponents at which all of the Puiseux expansions separate.) But if there is a multiple factor, then it is necessary to examine the situation more closely. For this, we turn to the Puiseux expansions.

Let us note that we will classify the real singular points. (It is possible for a real sextic curve to have a complex conjugate pair of singular points. We will avoid this case.) By a simple translation of axes, we may assume that the singular point is at the origin. Note that the notion of irreducible curve is with respect to the complex numbers. We may assume that no tangent line to the curve at the origin is vertical.

As an example, let a sextic curve be given by $f(x, y) = 0$, where

$$\begin{aligned} f(x, y) = {} & a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y \\ & + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 + a_{50}x^5 \\ & + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5 + a_{60}x^6 + a_{51}x^5y + a_{42}x^4y^2 \\ & + a_{33}x^3y^3 + a_{24}x^2y^4 + a_{15}xy^5 + a_{06}y^6. \end{aligned}$$

Since we may assume that our singular point is at $(0, 0)$, thus $a_{00} = 0$. Since the point is singular, thus $a_{10} = a_{01} = 0$. In this paper we will use the term "tangent cone" to refer to the terms of lowest degree in $f(x, y)$. The degree of these terms is called the multiplicity of the point. If the point is of multiplicity six, then the curve must be reducible since any homogeneous polynomial of degree 6 must factor. Thus, for irreducible curves of degree six, we only need to study points of multiplicity five, four, three, or two.

Let us now give a rough sketch of how all of the cases are enumerated. First we choose the tangent cone by choosing the tangent lines together with their multiplicities. For each tangent cone, we consider all possible Newton polygons. For each Newton polygon, we first consider the case where none of the quasihomogeneous polynomials corresponding to the

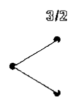
segments of the Newton polygon have a multiple factor. Then we consider the cases where there is a multiple factor. When there is a multiple factor, there are several cases where Maple is used to save many hours of hand calculation to compute the Puiseux expansion of the corresponding families of curves. The different types of singular points are then determined by the vanishing or nonvanishing of certain polynomials in the coefficients of the families of curves; these polynomials are given to us by the Maple computations. The details of this are carried out in the next section.

Let us now discuss the issue of verifying the existence of irreducible curves that have a given type of singular point. Observe that for a given degree, the irreducible curves form a dense open subset in the Zariski topology on the space of all curves of that degree. When a segment of the Newton polygon contains a multiple quasihomogeneous factor, we use Maple to determine the different types of singular points corresponding to that family, and in this process, a sequence of polynomial conditions on the coefficients (which turn out to be discriminants) is obtained. With respect to the Zariski topology, if an irreducible curve is found at any stage of the sequence, then all prior stages contain irreducible curves. Most of the time it is obvious that a certain family contains an irreducible representative. If it is not obvious, then Groebner basis techniques can be used to show that there is an irreducible representative or that every curve in a given family is reducible (even when Maple will not show this in response to the 'factor' command). (For more details on this see [9].)

Given an algebraic curve with a singular point at the origin, let us now describe how to associate a tree diagram to this singular point once we have the Puiseux expansions. Each time at least one "branch" separates, record the exponent where that happens. Place all such exponents in a row at the top. For each exponent in the top row, there corresponds a column of vertices. Each Puiseux expansion corresponds to exactly one vertex in that column, and those expansions with the same coefficients up to that exponent correspond to the same vertex. We start with one vertex on the left corresponding to the power zero. Braces will join pairs of vertices, within a given column, corresponding to complex conjugate coefficients. In such a case the only real solution of the original equation satisfying the pair of expansions indicated by the braces, in a small enough neighborhood of the origin is (0, 0). Line segments are drawn connecting the vertices from left to right, where each polygonal path from left to right corresponds to Puiseux expansions having the same set of coefficients up to a given exponent. The diagram stops at the first exponent where each vertex in that column corresponds to exactly one Puiseux expansion. Notice that this tree diagram uniquely specifies the singularity type (up to permutations of vertices within columns) provided that no tangent line at the origin is vertical.

Example $y^2 = -x^3$.

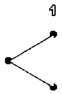
Notice that $y = \pm i\, x^{3/2}$, which can also be written as $y = \pm(-x)^{3/2}$. For each $x < 0$, there are two distinct real solutions for y. Hence, the diagram is (without braces!)



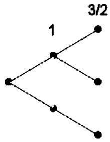
Example $x^2y + x^4 + 2xy^2 + y^3 = 0$. If $B := x^2y + x^4 + 2xy^2 + y^3$, the Maple command puiseux $(B, x = 0, y, 3)$ tells us that the Puiseux expansions begin as follows:

$$y = -x + x^{3/2} \quad \text{(branch\#1)}$$
$$y = -x - x^{3/2} \quad \text{(branch\#2)}$$
$$y = -x^2 \quad \text{(branch\#3)}$$

In the next section, we will refer to the relevant truncated portion of the Puiseux expansion as the *Puiseux jet*. Notice that the coefficient of x in branch #1 and branch #2 is -1, while the coefficient of x in branch #3 is 0. So there is a splitting at the first power of x, which is indicated as



Next we must show the splitting of #1 from #2. Notice that the power of x at which #1 and #2 split is $3/2$. Now our diagram looks like



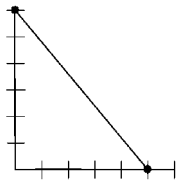
The diagram is now complete; notice that there are three distinct vertices in the column labeled $3/2$.

2 Classification and Proof

Irreducible curves

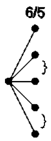
Multiplicity 5

Tangent cone: y^5.

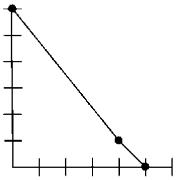
Newton polygon: 

$y^5 + ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6 = 0,\ a \neq 0.$

Puiseux jets:
$y = (-a)^{1/5}x^{6/5}$.

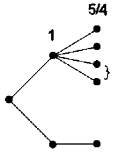
1. Diagram: 

Tangent cone: $y^4(y-x)$.

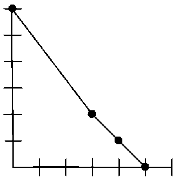
Newton polygon: 

$y^4(y-x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0$, $a\neq 0$.

Puiseux jets:
$y=x$
$y=a^{1/4}x^{5/4}$.

2. Diagram: 

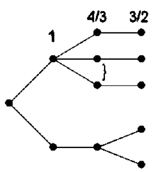
Tangent cone: $y^3(y-x)^2$.

Newton polygon: 

$y^3(y-x)^2+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0$, $a\neq 0$.

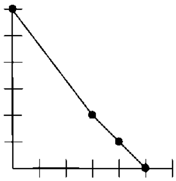
Condition: $a+b+c+d+e+f+g\neq 0$.

Puiseux jets:
$y=(-a)^{1/3}x^{4/3}$
$y=x-(a+b+c+d+e+f+g)^{1/2}x^{3/2}$.

3. Diagram: 

If $a+b+c+d+e+f+g=0$, then each curve in the resulting family has the factor $y-x$, so is reducible.

Tangent cone: $y^3(y-x)(y-2x)$.

Newton polygon: 

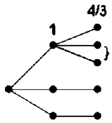
$$y^3(y-x)(y-2x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0,\ a\neq 0.$$

Puiseux jets:

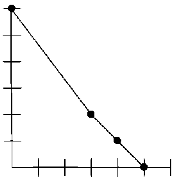
$y=(-\frac{a}{2})^{1/3}x^{4/3}$

$y=x$

$y=2x$.

4. Diagram: 

Tangent cone: $y^3(y^2+x^2)$.

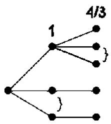
Newton polygon: 

$$y^3(y^2+x^2)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0,\ a\neq 0.$$

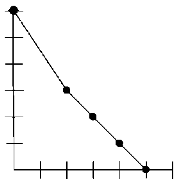
Puiseux jets:

$y=\pm ix$

$y=(-a)^{1/3}x^{4/3}$.

5. Diagram: 

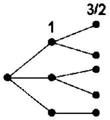
Tangent cone: $y^2(y-x)^2(y-2x)$.

Newton polygon: 

$y^2(y-x)^2(y-2x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0, a\neq 0.$

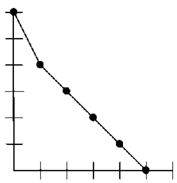
Condition: $a+b+c+d+e+f+g\neq 0$.

Puiseux jets:
$y=2x$
$y=(\frac{a}{2})^{1/2}x^{3/2}$
$y=x+(a+b+c+d+e+f+g)^{1/2}x^{3/2}$.

6. Diagram: 

If $a+b+c+d+e+f+g=0$, then each curve in the family has a factor of y–x, so is reducible.

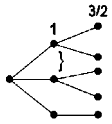
Tangent cone: $y(x^2+y^2)^2$.

Newton polygon: 

$y(x^2+y^2)^2+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0, a\neq 0.$

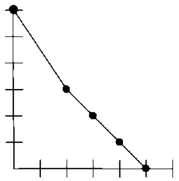
Condition: $-a+c-e+g\neq 0$ or $b+f-d\neq 0$.

Puiseux jets:
$y=0$
$y=\mathrm{RootOf}(1+Z^2)x+(\frac{1}{4}(\mathrm{RootOf}(1+Z^2)(-a+c-e+g)+(b+f-d)))^{1/2}x^{3/2}$.

7. Diagram: 

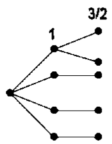
If $-a+c-e+g=0$ and $b+f-d=0$, then each curve in the family has a factor of x^2+y^2, so is reducible.

Tangent cone: $y^2(y-x)(y-2x)(y-3x)$.

Newton polygon: 

$$y^2(y-x)(y-2x)(y-3x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0,\ a\neq 0.$$

Puiseux jets:
$y=x$
$y=2x$
$y=3x$
$y=(\frac{a}{6})^{1/2}x^{3/2}$.

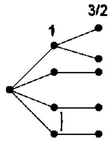
8. Diagram: 

Tangent cone: $y^2(y^2+x^2)(y-x)$.

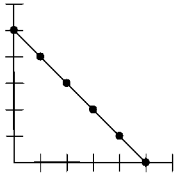
Newton polygon: 

$$y^2(y^2+x^2)(y-x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0,\ a\neq 0.$$

Puiseux jets:
$y=x$
$y=\pm ix$
$y=a^{1/2}x^{3/2}$.

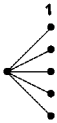
9. Diagram: 

Tangent cone: $(y-x)(y-2x)(y-3x)(y-4x)(y-5x)$.

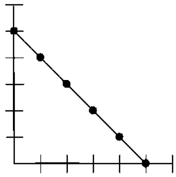
Newton polygon: 

$(y-x)(y-2x)(y-3x)(y-4x)(y-5x)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0$.

Puiseux jets:
$y=x$
$y=2x$
$y=3x$
$y=4x$
$y=5x$.

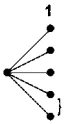
10. Diagram: 

Tangent cone: $(y-x)(y-2x)(y-3x)(y^2+x^2)$.

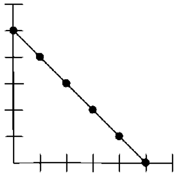
Newton polygon: 

$(y-x)(y-2x)(y-3x)(y^2+x^2)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0$.

Puiseux jets:
$y=x$
$y=2x$
$y=3x$
$y=\pm ix$.

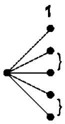
11. Diagram: 

Tangent cone: $(y-x)(y^2+x^2)(y^2+4x^2)$.

Newton polygon: 

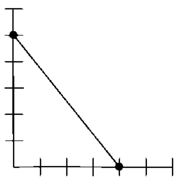
$$(y-x)(y^2+x^2)(y^2+4x^2)+ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6=0.$$

Puiseux jets:
$y=x$
$y=\pm ix$
$y=\pm 2ix$.

12. Diagram: 

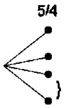
Multiplicity 4

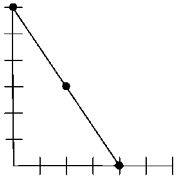
Tangent cone: y^4.

Newton polygon: 

$$y^4+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^6+hx^5y+jx^4y^2+kx^3y^3+lx^2y^4+mxy^5+ny^6=0,\ a\neq 0.$$

Puiseux jets:
$y=(-a)^{1/4}x^{5/4}$.

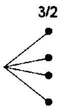
13. Diagram: 

Newton polygon: 

Quasihomogeneous factors: $(y^2 - x^3)(y^2 - 4x^3)$.

$(y - x^3)(y^2 - 4x^3) + ax^2y^3 + bxy^4 + cy^5 + dx^5y + ex^4y^2 + fx^3y^3 + gx^2y^4 + hxy^5 + jy^6 = 0$.

Puiseux jets:
$y = x^{3/2}$
$y = 2x^{3/2}$.

14. Diagram: 

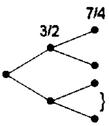
Quasihomogeneous factors: $(y^2 + x^3)^2$.

$(y^2 + x^3)^2 + ax^2y^3 + bxy^4 + cy^5 + dx^5y + ex^4y^2 + fx^3y^3 + gx^2y^4 + hxy^5 + jy^6 = 0$.

Maple gives us:

Puiseux jets:
$y = -(-x)^{3/2} + \frac{1}{2}(a - d)^{1/2}(-x)^{7/4}$.

Condition: $d \neq a$.

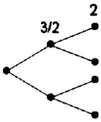
15. Diagram: 

Case: $d = a$.

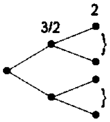
Puiseux jets:
$y = (-x)^{3/2} + x^2\mathrm{RootOf}(4Z^2 + 2aZ + e - b)$.

Condition: $a^2 - 4e + 4b \neq 0$.

Diagrams:

16. 

$a^2 - 4e + 4b > 0$

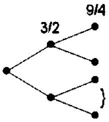
17. 

$a^2 - 4e + 4b < 0$

Case: $e = \frac{1}{16}(4a^2 + 16b)$.

Puiseux jets:
$y = -(-x)^{3/2} - \frac{a}{4}x^2 - \frac{1}{4}(2ab - 4f + 4c)^{1/2}(-x)^{9/4}$.

Condition: $ba - 2f + 2c \neq 0$.

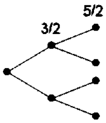
18. Diagram: 

Case: $f = \frac{1}{2}(ba + 2c)$.

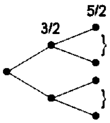
Puiseux jets:
$y = (-x)^{3/2} - \frac{a}{4}x^2 + (-x)^{5/2}\mathrm{RootOf}(1024Z^2 + (-512b - 64a^2)Z + (16ba^2 - 128ca + 256g + a^4))$.

Condition: $b^2 + 2ca - 4g \neq 0$.

Diagrams:

19. 

$b^2 + 2ca - 4g > 0$

20. 

$b^2 + 2ca - 4g < 0$

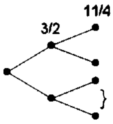
Case: $g = \frac{1}{4}(b^2 + 2ac)$.

Puiseux jets:

$y=(-x)^{3/2}-\frac{a}{4}x^2-\frac{a^2+8b}{32}(-x)^{5/2}+\frac{(-4h+2cb)^{1/2}}{4}(-x)^{11/4}.$

Condition: $cb-2h\neq 0.$

21. Diagram:



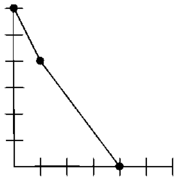
Case: $h=\frac{1}{2}bc.$

Puiseux jets:
$y=(-x)^{3/2}-\frac{a}{4}x^2-\frac{a^2+8b}{32}(-x)^{5/2}-x^3\mathrm{RootOf}(64Z^2+(32c+16ab)Z+4abc+16j+a^2b^2).$

Condition: $c^2-4j\neq 0.$

Groebner basis techniques show that each curve in the resulting family is reducible (even though Maple will not show this in response to the ‘factor’ command).

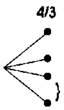
Newton polygon:



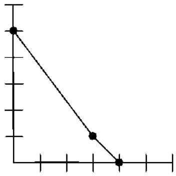
$y^4+ax^6+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^5y+hx^4y^2+jx^3y^3+kx^2y^4+lxy^5+my^6=0.$

Puiseux jets:
$y=-\frac{a}{b}x^2$
$y=(-b)^{1/3}x^{4/3}.$

22. Diagram:

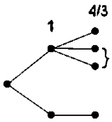


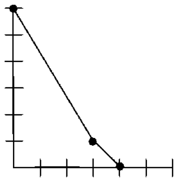
Tangent cone: $y^3(y-x).$

Newton polygon: 

$y^3(y-x)+ax^5+bx^4+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^6+hx^5y+jx^4y^2+kx^3y^3+lx^2y^4+mxy^5+ny^6$, $a \neq 0$.

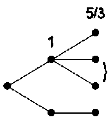
Puiseux jets:
$y = x$
$y = a^{1/3}x^{4/3}$.

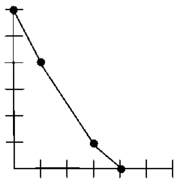
23. Diagram: 

Newton polygon: 

$y^3(y-x)+ax^6+bx^3y^2+cx^2y^3+dxy^4+ey^5+fx^5y+gx^4y^2+hx^3y^3+jx^2y^4+kxy^5+ly^6$, $a \neq 0$.

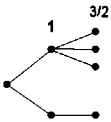
Puiseux jets:
$y = x$
$y = a^{1/3}x^{5/3}$.

24. Diagram: 

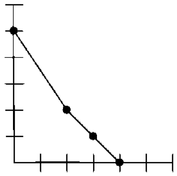
Newton polygon: 

$y^3(y-x)+ax^6+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^5y+hx^4y^2+jx^3y^3+kx^2y^4+lxy^5+my^6=0$, $a \neq 0$, $b \neq 0$.

Puiseux jets:
$y = x$
$y = -\frac{a}{b}x^2$
$y = b^{1/2}x^{3/2}$.

25. Diagram: 

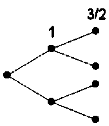
Tangent cone: $y^2(y-x)^2$.

Newton polygon: 

$y^2(y-x)^2 + ax^5 + bx^4 + cx^3y^2 + dx^2y^3 + exy^4 + fy^5 + gx^6 + hx^5y + jx^4y^2 + kx^3y^3 + lx^2y^4 + mxy^5 + ny^6$, $a \neq 0$.

Puiseux jets:
$y = a^{1/2}(-x)^{3/2}$
$y = x + (a+b+c+d+e+f)^{1/2}(-x)^{3/2}$.

Condition: $a+b+c+d+e+f \neq 0$.

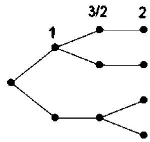
26. Diagram: 

Case: $f = -(a+b+c+d+e)$.

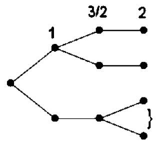
Puiseux jets:
$y = a^{1/2}(-x)^{3/2}$
$y = x + x^2\mathrm{RootOf}(Z^2 + (-5a-3c-e-4b-2d)Z + g+k+m+h+j+l+n)$.

Condition: $D_1 = 10ae + 40ab + 20ad + 9c^2 + 6ce + 24bc + 12cd + e^2 + 8be + 4de + 16bd + 4d^2 + 16b^2 + 30ac + 25a^2 - 4g - 4k - 4m - 4h - 4j - 4l - 4n \neq 0$.

Diagrams:

27. 

$D_1 > 0$

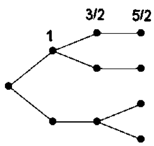
28. 

$D_1 < 0$

Case: $D_1 = 0$; solve for n.

From now on, the Puiseux jets will be too large to exhibit in this paper, so we will just exhibit the diagrams. The conditions are obtained after Maple computations of the Puiseux jets; these conditions are obtained by Maple computation of discriminants coming from these jets. During the course of these computations, Maple sometimes miraculously factors the polynomial discriminants or families of curves, giving dozens of interesting identities. Unfortunately, these identities are too long to exhibit in this paper, but the reader can see them on the Maple worksheets in the technical reports [9]. We will use the notation D_k to denote the factors of the discriminants that arise, but we will not always display these factors because they are too big.

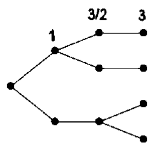
Condition: $D_2 D_3 = (3c + 2d + 4b + e + 5a)(3ec + 6eb + 10ea + ed - 4l + 50a^2 + 9c^2 + 24b^2 - 2m - 6k - 8j - 10h - 12a + 2d^2 + 9cd + 25ad + 70ab + 45ac + 16bd + 30cb) \neq 0$.

29. Diagram: 

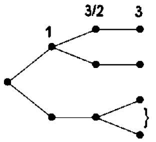
Case: $D_3 = 0$; solve for m.

Condition: $16{D_2}^2 D_4 = 16{D_2}^2(15c^2 + 130ac + 68cb + 10cd + 2ce + 20ae + 240ab + 60ad + 8eb + 28bd + d^2 - 40h - 4l - 60g + 68b^2 + 200a^2 - 12k - 24j) \neq 0$.

Diagrams:

30. 

$D_4 > 0$

31. 

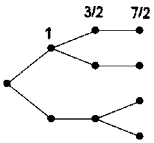
$D_4 < 0$

Case: $D_2 = 0$; solve for e.

Then each curve in the family is reducible with $(x - y)^2$ as a factor.

Case: $D_4 = 0$; solve for l.

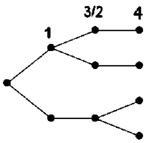
Condition: ${D_2}^3 D_5 = {D_2}^3(3c^2 + 55ac + cd + 21cb + eb + 20ad + 6bd + 5ae - 20h - 8j + 28b^2 + 125a^2 - 40g - 2k - 125ab) \neq 0$.

32. Diagram: 

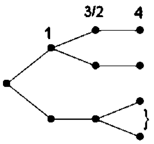
Case: $D_5 = 0$; solve for k.

Condition: $256{D_2}^4 D_6 = 256{D_2}^4(c^2 + 14cb + 56ac + 2bd + 28b^2 + 210a^2 + 168ab + 2ae + 14ad - 20h - 60g - 4j) \neq 0$.

Diagrams:

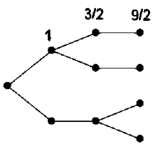
33. 

$D_6 > 0$

34. 

$D_6 < 0$

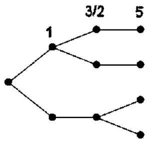
Case: $D_6 = 0$; solve for j. Condition: ${D_2}^5 D_7 = {D_2}^5(8ac + cb + 60a^2 + 4b^2 + ad + 36ab - 2h - 12g) \neq 0$.

35. Diagram: 

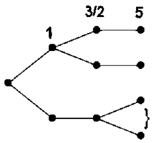
Case: $D_7 = 0$; solve for h.

Condition: $1024{D_2}^6 D_8 = 1024{D_2}^6(2ac + 18ab + 45a^2 + b^2 - 4g) \neq 0$.

Diagrams:

36. 

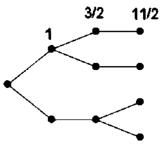
$D_8 > 0$

37. 

$D_8 < 0$

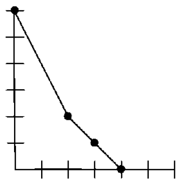
Case: $D_8 = 0$; solve for g.

Condition: ${D_2}^7(b + 5a) \neq 0$.

38. Diagram: 

Case: $b = -5a$.

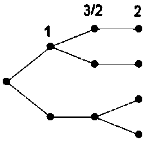
Then the resulting family is $\frac{1}{4}(ey^2 + cy^2 - 5ay^2 - 2y + dy^2 + 2x + cxy + dxy + cx^2 - 10ax^2)(cx^2y^2 + cxy^3 + cy^4 + 2ax^4 - 8ax^3y + 2ax^2y^2 + 2xy^2 + dxy^3 - 8axy^3 + dy^4 + ey^4 - 3ay^4 - 2y^3) = 0$.

Newton polygon: 

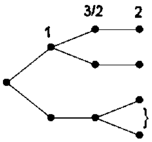
$y^2(y - x)^2 + lx^6 + mx^4y + ax^3y^2 + bx^2y^3 + dy^5 + ex^5y + fx^4y^2 + gx^3y^3 + hx^2y^4 + jxy^5 + ky^6 = 0$, $m^2 - 4l \neq 0$.

Condition: $a + b + c + d + m \neq 0$.

Diagrams:

27. 

$m^2 - 4l > 0$

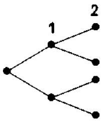
28. 

$m^2 - 4l < 0$

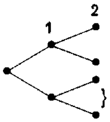
Case: $a = -b - c - d - m$.

Condition: $D_1 = -4l - 4j - 4f - 4g - 4h - 4e - 4k + 9d^2 + 12dc + 6db - 6dm + 4c^2 + 4cb - 4cm + b^2 - 2bm + m^2 \neq 0$.

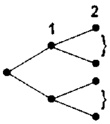
Diagrams:

39. 

$m^2 - 4l > 0$ and
$D_1 > 0$

40. 

$m^2 - 4l < 0$ and $D_1 > 0$
or $m^2 - 4l > 0$ and $D_1 < 0$

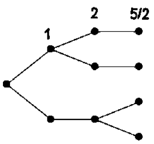
41. 

$m^2 - 4l < 0$ and
$D_1 < 0$

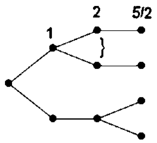
Case: $D_1 = 0$; solve for j.

Condition: $D_2 D_3 = (3d + 2c_b - m)(-3b^2 - 12db - 10cb + 8bm - 5m^2 + 8g - 9d^2 + 16e + 14cm + 20l + 4h + 12f - 8c^2 - 4k - 18dc + 18dm) \neq 0$.

Diagrams:

42. 

$m^2 - 4l > 0$

43. 

$m^2 - 4l < 0$

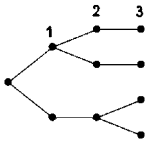
Case: $D_2 = 0$; solve for b.

Then each curve in the resulting family has a factor of $x - y$, hence is reducible. So $D_2 \neq 0$ in the rest of the cases.

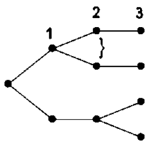
Case: $D_3 = 0$; solve for g.

Condition: $-128{D_2}^2 D_4 = -128{D_2}^2(-3b^2 - 6cb + 16bm - 4db - 4h + 32e + 12f - 3d^2 + 60l + 12k - 2dc - 15m^2 + 22dm - 2c^2 + 22cm) \neq 0$.

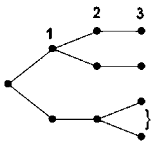
Diagrams:

44. 

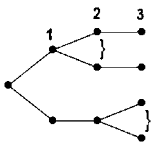
$m^2 - 4l > 0$ and
$D_4 < 0$

45. 

$m^2 - 4l < 0$ and
$D_4 > 0$

46. 

$m^2 - 4l > 0$ and
$D_4 > 0$

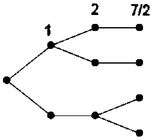
47. 

$m^2 - 4l < 0$ and
$D_1 > 0$

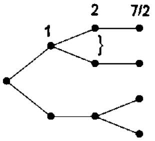
Case: $D_4 = 0$; solve for h.

Condition: ${D_2}^3 D_5 = {D_2}^3(-b^2 + 8bm - 2db - 2cb - c^2 - 2dc - 10m^2 + 16e - 4k + 10dm + 40l + 4f + 10cm) \neq 0$.

Diagrams:

48. 

$m^2 - 4l > 0$

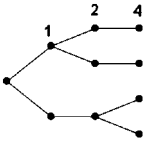
49. 

$m^2 - 4l < 0$

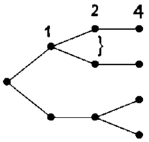
Case: $D_5 = 0$; solve for f.

Condition: $-256 {D_2}^4 D_6 = -256 {D_2}^4 (2bm - 5m^2 + 2dm - d^2 + 4e + 4k + 20l + 2cm) \neq 0.$

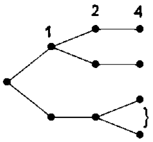
Diagrams:

50. 

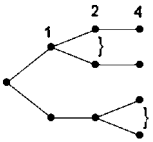
$m^2 - 4l > 0$ and $D_6 < 0$

51. 

$m^2 - 4l < 0$ and $D_6 < 0$

52. 

$m^2 - 4l > 0$ and $D_6 > 0$

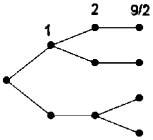
53. 

$m^2 - 4l < 0$ and $D_6 > 0$

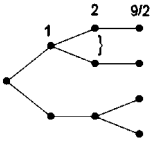
Case: $D_6 = 0$, solve for e.

Condition: ${D_2}^5 D_7 = {D_2}^5 (d^2 - 4k - m^2 + 4l) \neq 0.$

Diagrams:

54. 

$m^2 - 4l > 0$

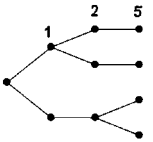
55. 

$m^2 - 4l < 0$

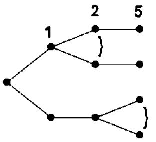
Case: $D_7 = 0$; solve for k.

Condition: $-1024{D_2}^6(4l - m^2) \neq 0$.

Diagrams:

56. 

$m^2 - 4l > 0$

57. 

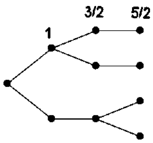
$m^2 - 4l < 0$

Groebner basis techniques show that the resulting family contains irreducible curves. Notice that in the family we started with, $m^2 - 4l \neq 0$, so we are done with this family. So now consider the family

$y^2(y-x)^2 + lx^6 + mx^4y + ax^3y^2 + bx^2y^3 + dy^5 + ex^5y + fx^4y^2 + gx^3y^3 + hx^2y^4 + jxy^5 + ky^6 = 0$, $m^2 - 4l = 0$, $m \neq 0$, $l \neq 0$.

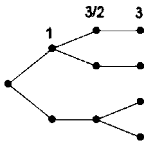
Case *1: $a + b + c + d + m \neq 0$.

Condition: $m^2 + am - 2e \neq 0$.

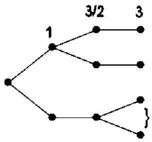
29. Diagram: 

Case: $m^2 + am - 2e = 0$; solve for e. Condition: $D_1 = 3m^2 + 2mb + 4am - 4f + a^2 \neq 0$.

Diagrams:

30. 

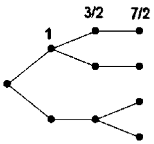
$D_1 > 0$

31. 

$D_1 < 0$

Case: $D_1 = 0$; solve for f.

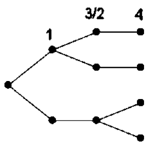
Condition: $D_2 = 3am + a^2 + 2m^2 + 2mb + ba - 2g + cm \neq 0$.

32. Diagram: 

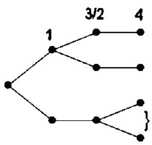
Case: $D_2 = 0$; solve for g.

Condition: $D_3 = 5m^2 + 6bm + 8am + 4cm + 2md + 2ac + b^2 + 3a^2 - 4h + 4ab \neq 0$.

Diagrams:

33. 

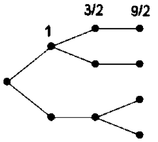
$D_3 > 0$

34. 

$D_3 < 0$

Case: $D_3 = 0$; solve for h.

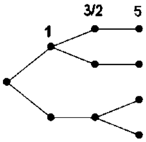
Condition: $D_4 = 3cm - 2j + 2a^2 + 5am + cb + 3ba + 3m^2 + b^2 + 4mb + da + 2ac + 2md \neq 0$.

35. Diagram: 

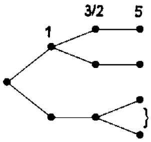
Case: $D_4 = 0$; solve for j.

Condition: $D_5 = 7m^2 + 12am + 8cm + 10mb + 6md + 2db + 3b^2 - 4k + 5a^2 + 4cb + c^2 + 4da + 8ba + 6ac \neq 0$.

Diagrams:

36. 

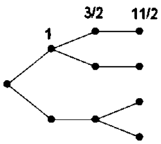
$D_5 > 0$

37. 

$D_5 < 0$

Case: $D_5 = 0$; solve for k.

Condition: $(4m + 2b + c + 3a)(c + m + b + d + a) \neq 0$.

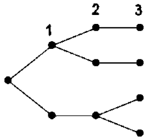
38. Diagram: 

In either of the cases $4m + 2b + c + 3a = 0$ or $c + m + b + d + a = 0$, each curve in the resulting family is reducible.

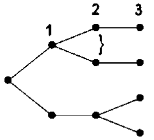
Case *2: $a + b + c + d + m = 0$; solve for a: and $D_6 = -64j - 64f - 64g - 64h - 64e - 64k + 144d^2 + 192dc - 96md + 96db + 64c^2 - 64cm + 64cb - 32mb + 16b^2 \neq 0$.

Condition: $D_7 = -2md - 2cm - 4f + d^2 + 2db + 2dc + b^2 + 2cb + c^2 \neq 0$.

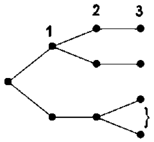
Diagrams:

44. 

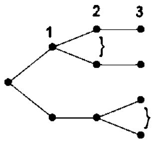
$D_6 > 0$ and $D_7 > 0$

45. 

$D_6 < 0$ and $D_7 > 0$

46. 

$D_6 > 0$ and $D_7 < 0$

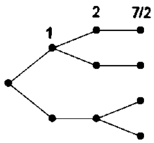
47. 

$D_6 < 0$ and $D_7 < 0$

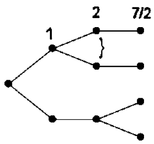
Case: $D_7 = 0$; solve for f.

Condition: $D_8 = -c^2 + md - d^2 + 2g - db - cb - 2dc \neq 0$.

Diagrams:

48. 

$D_6 > 0$

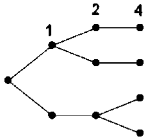
49. 

$D_6 < 0$

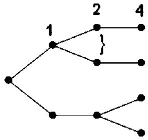
Case: $D_8 = 0$; solve for g.

Condition: $D_9 = -4h + 3d^2 + c^2 + 2db + 4dc \neq 0$.

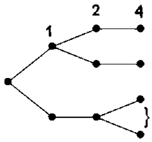
Diagrams:

50. 

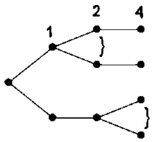
$D_6 > 0$ and $D_9 > 0$

51. 

$D_6 < 0$ and $D_9 > 0$

52. 

$D_6 > 0$ and $D_9 < 0$

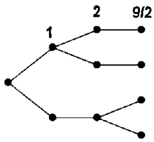
53. 

$D_6 < 0$ and $D_9 < 0$

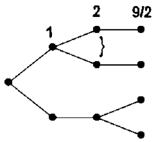
Case: $D_9 = 0$; solve for h.

Condition: $D_{10} = dc - 2j + d^2 \neq 0$.

Diagrams:

54. 

$D_6 > 0$

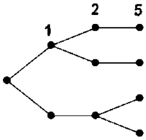
55. 

$D_6 < 0$

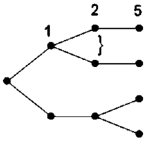
Case: $D_{10} = 0$; solve for j.

Condition: $d^2 - 4k \neq 0$.

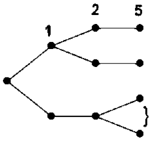
Diagrams:

56. 

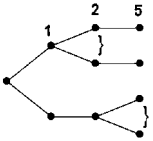
$D_6 > 0$ and $d^2 - 4k > 0$

58. 

$D_6 < 0$ and $d^2 - 4k > 0$

59. 

$D_6 > 0$ and $d^2 - 4k < 0$

57. 

$D_6 < 0$ and $d^2 - 4k < 0$

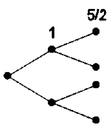
Groebner basis techniques show that the resulting family contains irreducible curves.

Case: $d^2 - 4k = 0$, solve for k.

Each curve in the resulting family is reducible.

Case *3: $D_6 = 0$, solve for e: and $\frac{1}{8}D_{11}D_{12} = \frac{1}{8}(2c + 3d - m + b)(-b^2 - 6cb - 12db + 12h - 30dc + 6md + 4f + 2cm + 20k + 16j - 8c^2 + 8g - 27d^2) \neq 0$.

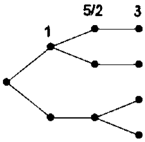
Condition: $D_{13} = 2cm - 9d^2 - 4c^2 + 4h + 4k - 12dc - 6db - 4cb + 4md + 4j + 4f + 4g - b^2 \neq 0$.

60. Diagram: 

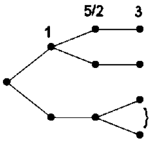
Case: $D_{13} = 0$; solve for j.

Condition: $D_{14} = -2md - 2cm - 4f + d^2 + 2db + 2dc + b^2 + 2cb + c^2 \neq 0$.

Diagrams:

61. 

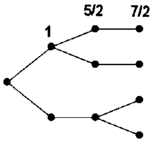
$D_{14} > 0$

62. 

$D_{14} < 0$

Case: $D_{14} = 0$; solve for f.

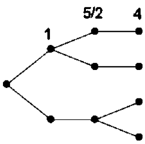
Condition: $D_{15} = -c^2 + md - d^2 + 2g - db - cb - 2dc \neq 0$.

63. Diagram: 

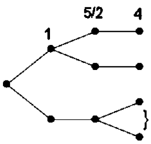
Case: $D_{15} = 0$; solve for g.

Condition: $D_{16} = -4h + 3d^2 + c^2 + 2db + 4dc \neq 0$.

Diagrams:

64. 

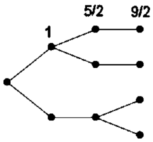
$D_{16} > 0$

65. 

$D_{16} < 0$

Case: $D_{16} = 0$; solve for h.

Condition: $d^2 - 4k \neq 0$.

66. Diagram: 

Case: $d^2 - 4k = 0$, solve for k.

Each curve in the resulting family is reducible.

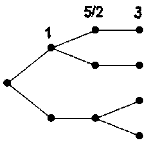
Case *4: $D_{11} = 0$; solve for m.

Each curve in the resulting family is reducible.

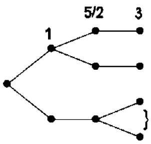
Case *5: $D_{12} = 0$; solve for f: and $-64D_{11}{}^2 D_{17} = -64D_{11}{}^2(-2cb - 8db + 2md + 40k - 28dc + 24j + 12h - 5c^2 - 33d^2 + 4g) \neq 0$.

Condition: $D_{18} = -9dc - 3db + 6j - cb - 9d^2 - 2c^2 + 2g + 4h + 8k + md \neq 0$.

Diagrams:

61. 

$D_{18} > 0$

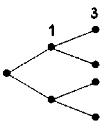
62. 

$D_{18} < 0$

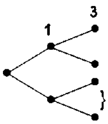
Case: $D_{18} = 0$; solve for g.

Condition: $D_{19} = 10d^2 - 4h - 12k - 8j + 8dc + 2db + c^2 \neq 0$.

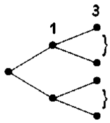
Diagrams:

67. 

$D_{17} < 0$ and $D_{19} > 0$

68. 

$D_{17} > 0$ and $D_{19} > 0$ or
$D_{17} < 0$ and $D_{19} < 0$

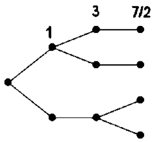
69. 

$D_{17} > 0$ and $D_{19} < 0$

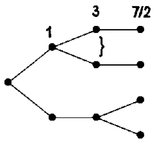
Case: $D_{19} = 0$; solve for h.

Condition: $D_{20} = 2d^2 - 2j - 4k + dc \neq 0$.

Diagrams:

70. 

$D_{17} < 0$

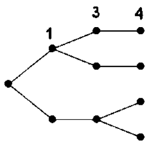
71. 

$D_{17} > 0$

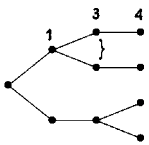
Case: $D_{20} = 0$; solve for j.

Condition: $d^2 - 4k \neq 0$.

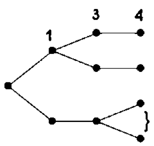
Diagrams:

72. 

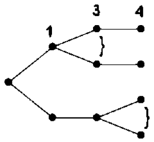
$D_{17} < 0$ and $d^2 - 4k > 0$

73. 

$D_{17} > 0$ and $d^2 - 4k > 0$

74. 

$D_{17} < 0$ and $d^2 - 4k < 0$

75. 

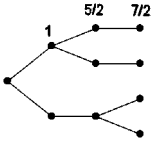
$D_{17} > 0$ and $d^2 - 4k < 0$

Case: $d^2 - 4k = 0$, solve for k.

Each curve in the resulting family is reducible.

Case *6: $D_{17} = 0$, solve for g: and $-\frac{1}{32}{D_{11}}^3 D_{21} = -\frac{1}{32}{D_{11}}^3(2db + 12dc + c^2 + 21d^2 - 4h - 16j - 40k) \neq 0$.

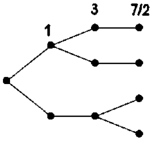
Condition: $D_{22} = -24k + 10dc - 4h + 15d^2 + c^2 - 12j + 2db \neq 0$.

63. Diagram: 

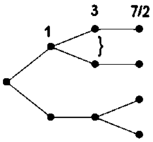
Case: $D_{22} = 0$; solve for h.

Condition: $-64m^2 D_{23} = -64m^2(5d^2 - 4j + 2dc - 12k) \neq 0$.

Diagrams:

70. 

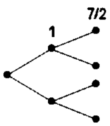
$D_{23} < 0$

71. 

$D_{23} > 0$

Case: $D_{23} = 0$; solve for j.

Condition: $d^2 - 4k \neq 0$.

76. Diagram: 

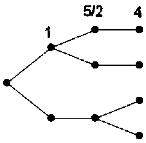
Case: $d^2 - 4k = 0$; solve for k.

Each curve in the resulting family is reducible.

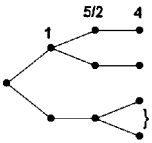
Case *7: $D_{21} = 0$; solve for h: and $D_{24} = -20k + 2dc - 4j + 7d^2 \neq 0$.

Condition: $D_{25} = dc - 2j + 3d^2 - 8k \neq 0$.

Diagrams:

64. 

$D_{25} > 0$

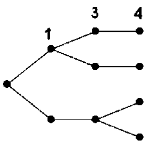
65. 

$D_{25} < 0$

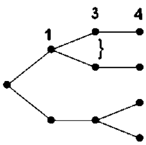
Case: $D_{25} = 0$; solve for j.

Condition: $d^2 - 4k \neq 0$.

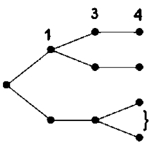
Diagrams:

72. 

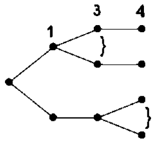
$D_{25} > 0$ and $d^2 - 4k > 0$

73. 

$D_{25} > 0$ and $d^2 - 4k < 0$

74. 

$D_{25} < 0$ and $d^2 - 4k > 0$

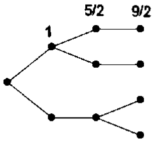
75. 

$D_{25} < 0$ and $d^2 - 4k < 0$

Case: $d^2 - 4k = 0$; solve for k.

Each curve in the resulting family is reducible.

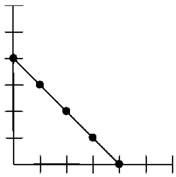
Case *8: $D_{24} = 0$; solve for j: and $d^2 - 4k \neq 0$.

66. Diagram: 

Case: $d^2 - 4k = 0$; solve for k.

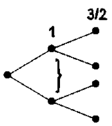
Each curve in the resulting family is reducible.

Tangent cone: $(x^2 + y^2)^2$.

Newton polygon: 

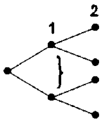
$(x^2+y^2)^2+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^6+hx^5y+jx^4y^2+kx^3y^3+lx^2y^4+mxy^5+ny^6=0.$

Condition: $e + a - c \neq 0$ or $f + b - d \neq 0$.

77. Diagram: 

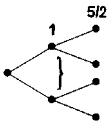
Case: $e + a - c \neq 0$ (solve for c) and $f + b - d = 0$ (solve for d).

Condition: $k \neq \frac{1}{16}(8fa - 8ba - 8fe + 8be + 16m + 16h)$ or $n \neq \frac{1}{16}(-4a^2 + 8ae - 4e^2 + 4f^2 - 8fb + 4b^2 + 16g - 16j + 16l)$.

78. Diagram: 

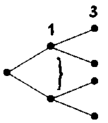
Case: $k = \frac{1}{16}(8fa - 8ba - 8fe + 8be + 16m + 16h)$ and $n = \frac{1}{16}(-4a^2 + 8ae - 4e^2 + 4f^2 - 8fb + 4b^2 + 16g - 16j + 16l)$.

Condition: $D_1 = -4abf - 8ja + 2ef^2 + 4ab^2 - 2eb^2 + 4nb - 12ge + 12ga + 4hb + 4hf - 5e^2a + e^3 + 7a^2e + 8je + 4la - 4le - 4mf - 3a^2 \neq 0$,
or $D_2 = 2b^3 + 4me - 4ma - 4he + 4ha - 4lf - 2aef + 6aeb + 8jf + 3a^2f + 4lb - e^2b - 5a^2b - 12gf - 8jb - 4fb^2 + 2f^2b - e^2f + 12gb \neq 0$.

79. Diagram: 

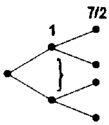
Case: $D_1 = 0$ and $D_2 = 0$, solve for m in both. Set the two solutions for m equal to each other, and solve this equation for l. (If it is not possible to solve for m, i.e. if $a = e$ and $b = f$, then nothing new is obtained.)

Condition: $j \neq \frac{1}{4}(-3a^2 + 2ae + b^2 + 12g)$ or $h \neq \frac{1}{2}ab$.

80. Diagram: 

Case: $j = \frac{1}{4}(-3a^2 + 2ae + b^2 + 12g)$ and $h = \frac{1}{2}ab$.

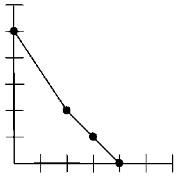
Condition: $4g - a^2 \neq 0$.

81. Diagram: 

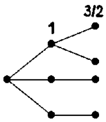
Case: $4g - a^2 = 0$; solve for g.

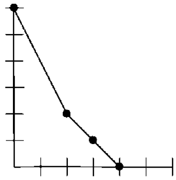
Each curve in the resulting family is reducible.

Tangent cone: $y^2(y - x)(y - 2x)$.

Newton polygon: 

$y^2(y-x)(y-2x)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^6+hx^5y+jx^4y^2+kx^3y^3+lx^2y^4+mxy^5+ny^6=0$, $a\neq 0$.

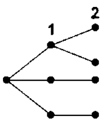
82. Diagram: 

Newton polygon: 

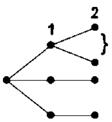
$y^2(y-x)(y-2x)+2lx^4y+mx^6+ax^3y^2+bx^2y^3+cxy^4+dy^5+ex^5y+fx^4y^2+gx^3y^3+hx^2y^4+jxy^5+ky^6=0$, $m\neq 0$.

Condition: $4l^2-8m\neq 0$.

Diagrams:

83. 

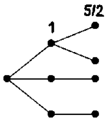
$4l^2-8m>0$

84. 

$4l^2-8m<0$

Case: $4l^2-8m=0$, solve for m.

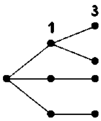
Condition: $3l^2+2al-4e\neq 0$.

85. Diagram: 

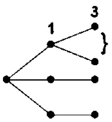
Case: $3l^2 + 2al - 4e = 0$; solve for e.

Condition: $D_1 = 4lb + 7l^2 - 8f + a^2 + 6al \neq 0$.

Diagrams:

86. 

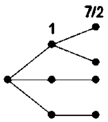
$D_1 > 0$

87. 

$D_1 < 0$

Case: $D_1 = 0$; solve for f.

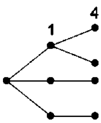
Condition: $D_2 = 14al + 15l^2 + 3a^2 + 4ba + 12lb - 16g + 8cl \neq 0$.

88. Diagram: 

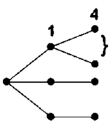
Case: $D_2 = 0$; solve for g.

Condition: $D_3 = 16ld + 12ab + 28lb + 8ca - 32h + 7a^2 + 30al + 24cl + 3l^2 + 4b^2 \neq 0$.

Diagrams:

89. 

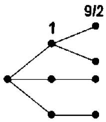
$D_3 > 0$

90. 

$D_3 < 0$

Case: $D_3 = 0$; solve for h.

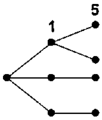
Condition: $D_4 = 16da - 64j + 12b^2 + 15a^2 + 62al + 28ba + 48ld + 63l^2 + 60lb + 16cb + 24ca + 56cl \neq 0$.

91. Diagram: 

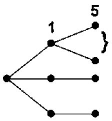
Case: $D_4 = 0$; solve for j.

Condition: $D_5 = -128k + 31a^2 + 126al + 28b^2 + 124lb + 60ba + 112ld + 48cb + 56ca + 120cl + 127l^2 + 32db + 48da + 16c^2 \neq 0$.

Diagrams:

92. 

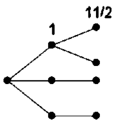
$D_5 > 0$

93. 

$D_5 < 0$

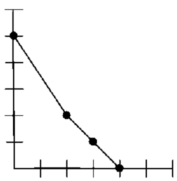
Case: $D_5 = 0$; solve for k.

Condition: $D_6D_7 = (15l + 7a + 6b + 4c)(9a + 10b + 12c + 16d + 17l) \neq 0$.

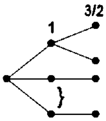
94. Diagram: 

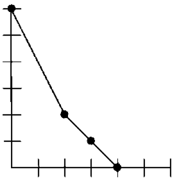
If $D_6 = 0$ or $D_7 = 0$, then each curve in the resulting family is reducible.

Tangent cone: $y^2(y^2 + x^2)$.

Newton polygon: 

$y^2(y^2+x^2)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5+gx^6+hx^5y+jx^4y^2+kx^3y^3+lx^2y^4+mxy^5+ny^6=0$, $a \neq 0$.

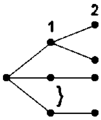
95. Diagram: 

Newton polygon: 

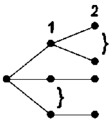
$y^2(y^2+x^2)+2lx^4y+mx^6+ax^3y^2+bx^2y^3+cxy^4+dy^5+ex^5y+fx^4y^2+gx^3y^3+hx^2y^4+jxy^5+ky^6=0.$

Condition: $l^2-m\neq 0.$

Diagrams:

96. 

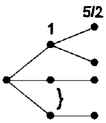
$l^2-m>0$

97. 

$l^2-m<0$

Case: $l^2-m=0$; solve for m.

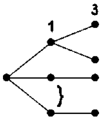
Condition: $al-e\neq 0.$

98. Diagram: 

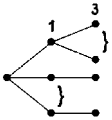
Case: $al-e=0$; solve for e.

Condition: $-l^2D_1=-l^2(4f-4lb+4l^2-a^2)\neq 0.$

Diagrams:

99. 

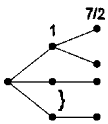
$D_1 < 0$

100. 

$D_1 > 0$

Case: $D_1 = 0$; solve for f.

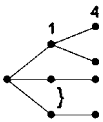
Condition: $D_2 = -ba + 2g + 2al - 2cl \neq 0$.

101. Diagram: 

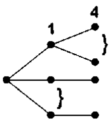
Case: $D_2 = 0$; solve for g.

Condition: $-256l^4 D_3 = -256l^4(-4l^2 - 4ld + 4lb + a^2 - 2ac + 4h - b^2) \neq 0$.

Diagrams:

102. 

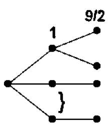
$D_3 < 0$

103. 

$D_3 > 0$

Case: $D_3 = 0$; solve for h.

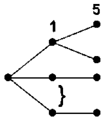
Condition: $D_4 = cb + 2al - 2j - ba + da - 2cl \neq 0$.

104. Diagram: 

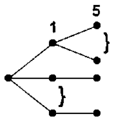
Case: $D_4 = 0$; solve for j.

Condition: $-4096l^6 D_5 = -4096l^6(-c^2+4k+b^2+4l^2+2ac-2db-a^2-4lb+4ld) \neq 0$.

Diagrams:

105. 

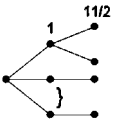
$D_5 < 0$

106. 

$D_5 > 0$

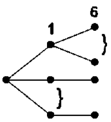
Case: $D_5 = 0$; solve for k.

Condition: $(a-c)(-b+d+2l) \neq 0$.

107. Diagram: 

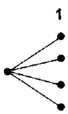
If $a = c$, then each curve in the resulting family is reducible.

If $b = d + 2l$, and $a \neq c$ then Groebner basis techniques show that the resulting family contains irreducible curves.

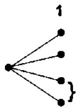
108. Diagram: 

Then, if $a = c$, each curve in the resulting family is reducible.

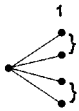
Tangent cone: $(y-x)(y-2x)(y-3x)(y-4x)$.

109. Diagram: 

Tangent cone: $(y-x)(y-2x)(x^2+y^2)$.

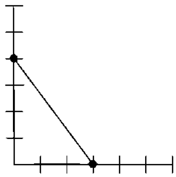
110. Diagram: 

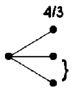
Tangent cone: $(y^2+x^2)(y^2+4x^2)$.

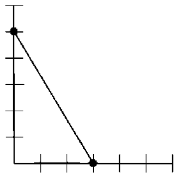
111. Diagram: 

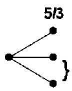
Multiplicity 3

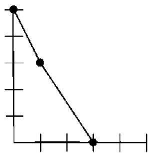
Tangent cone: y^3.

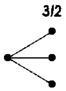
Newton polygon: 

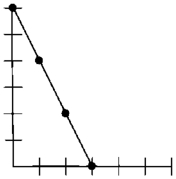
112. Diagram: 

Newton polygon: 

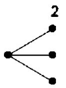
113. Diagram: 

Newton polygon: 

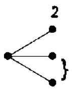
114. Diagram: 

Newton polygon: 

Quasihomogeneous factors: $(y-ax^2)(y-bx^2)(y-cx^2)$, a, b, c real and distinct.

115. Diagram: 

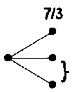
Quasihomogeneous factors: $(y-ax^2)(y-bx^2)(y-cx^2)$, a real, b and c are complex conjugate.

116. Diagram: 

Quasihomogeneous factors: $(y-x^2)^3$.

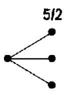
$(y-x)^3+ax^5y+bx^3y^2+cxy^3+dx^4y^2+ex^2y^3+fy^4+gx^3y^3+hxy^4+jx^2y^4+ky^5+lxy^5+my^6=0.$

Condition: $a+b+c\neq 0$.

117. Diagram: 

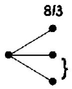
Case: $a+b+c=0$, solve for a.

Condition: $b+2c\neq 0$.

118. Diagram: 

Case: $b+2c=0$, solve for b.

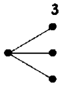
Condition: $d+e+f \neq 0$.

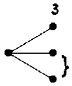
119. Diagram: 

Case: $d+e+f=0$; solve for e.

Condition: $D_1 = c^2d^2 - 18cdg - 18cdh - 2c^2df - 27g^2 - 54gh + 18gcf - 27h^2 + 18hcf + c^2f^2 - 4c^3g - 4c^3h - 12fd^2 + 12f^2d - 4f^3 + 4d^3 \neq 0$.

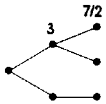
Diagrams:

120. 

121. 

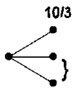
Case: $D_1 = 0$; solve for h.

Condition: $(-3d+3f-c^2)^2 \neq 0$.

122. Diagram: 

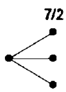
Case: $f = \frac{1}{3}(3d+c^2)$.

Condition: $D_2 = 2c^4 + 9c^2d + 27gc + 81j + 81k \neq 0$.

123. Diagram: 

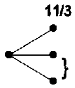
Case: $D_2 = 0$; solve for k.

Condition: $\frac{2}{3}cd + \frac{5}{27}c^3 + g \neq 0$.

124. Diagram: 

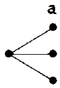
Case: $g = -\frac{2}{3}cd - \frac{5}{27}c^3$.

Condition: $-l + \frac{c^3d}{27} + \frac{c^5}{81} - \frac{jc}{3} \neq 0$.

125. Diagram: 

Case: $l = \frac{c^3 d}{27} + \frac{c^5}{81} - \frac{jc}{3}$.

Condition: $D_3 \neq 0$. (D_3 is not shown here because it is fairly long. We refer to the Maple worksheet in [9].)

126. Diagram: $(a = 4)$ 

Case: $D_3 = 0$, solve for m.

Groebner basis techniques show that each curve in the resulting family is reducible (even though Maple will not show this in response to the ‘factor’ command).

We now show more details than usual because it is necessary to combine Maple calculations with special observations.

$$B = -\frac{y^6c^2d^2}{27} - \frac{y^6jd}{3} - \frac{y^6jc^2}{9} + \frac{xy^5c^5}{81} + \frac{y^5c^2d}{9} + \frac{2xy^4c^3}{9} - \frac{5x^3y^3c^3}{27} - 2x^2y^3d - \frac{x^2y^3c^2}{3} - 2cx^3y^2 + 2xy^4cd - 2x^3y^3cd + \frac{y^4c^2}{3} + y^4d - \frac{xy^5jc}{3} + \frac{xy^5c^3d}{27} - \frac{2y^6d^3}{27} + \frac{y^6c^6}{729} - y^5j + \frac{y^5c^4}{27} + cxy^3 + dx^4y^2 + jx^2y^4 + y^3 - x^6 - 3y^2x^2 + 3yx^4 + yx^5c + \frac{2y^6 3^{1/2}(3d^2+c^2d+9j)^{3/2}}{243} = 0.$$

Maple will not factor B, but it will factor $B - \frac{2y^6 3^{1/2}(3d^2+c^2d+9j)^{3/2}}{243}$ as

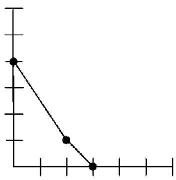
$$-\frac{1}{729}(9x^2 - 3cxy - 3dy^2 - 9y - c^2y^2)(81x^4 - 54x^3yx - 162x^2y - 9c^2x^2y^2 - 54dx^2y^2 + 6c^3xy^3 + 54cxy^2 + 18cdxy^3 - 81jy^4 + 54dy^3 + 81y^2 - 3c^2dy^4 - 18d^2y^4 + c^4y^4 + 18c^2y^3).$$

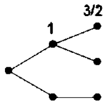
We observe that the latter is

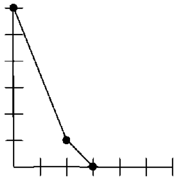
$$\frac{1}{729}[-(9x^2 - 3ycx - 3dy^2 - 9y - c^2y^2)^3 - (9x^2 - 3ycx - 3dy^2 - 9y - c^2y^2)(-81j - 3c^2d - 18d^2)y^4].$$

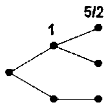
Therefore, B is a homogeneous polynomial in $x^2 - 3ycx - 3dy^2 - 9y - c^2y^2$ and y^2, so each curve in the family given by B is reducible!

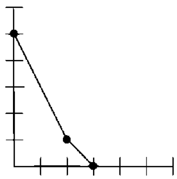
Tangent cone: $y^2(x - ay)$, $a \neq 0$.

Newton polygon: 

127. Diagram: 

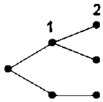
Newton polygon: 

128. Diagram: 

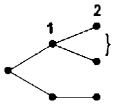
Newton polygon: 

Quasihomogeneous factors: $x(y - bx^2)(y - cx^2)$.

Diagrams:

129. 

b and c real and distinct

130. 

b and c complex conjugate

Quasihomogeneous factors: $x(y - x^2)^2$.

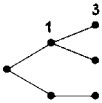
We wish to consider the general family $y^2(x - ay) - 2x^3y + x^5 + bx^6 + cx^4y + rx^2y^2 + dx^5y + ex^3y^2 + fxy^3 + gx^4y^2 + hx^2y^3 + jy^4 + kx^3y^3 + lxy^4 + mx^2y^4 + ny^5 + pxy^5 + qy^6 = 0$.

It turns out that Maple computation with this family is too big for our computer. The singular points at the origin for the curves in this family belong to the D_k series. By the theorem of Gudkov [4, p. 2], the maximum possible k for an irreducible sextic curve is 19. By Maple computation [9], the following family can be seen to have all D_k singular points for $8 \le k \le 19$.

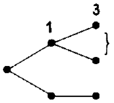
$y^2(x-ay)-2x^3y+x^5+a(x^6-3x^4y+3x^2y^2)+dx^5y+ex^3y^2+fxy^3+gx^4y^2+hx^2y^3+jy^4+kx^3y^3+lxy^4+mx^2y^4+ny^5+pxy^5+qy^6=0.$

Condition: $d+e+f \neq 0.$

Diagrams:

131. 

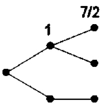
$d+e+f>0$

132. 

$d+e+f<0$

Case: $e=-f-d.$

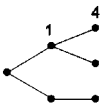
Condition: $g+h+j \neq 0.$

133. Diagram: 

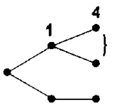
Case: $j=-h-g.$

Condition: $D_1=-4k-4l+f^2-2fd+d^2 \neq 0.$

Diagrams:

134. 

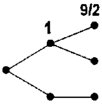
$D_1>0$

135. 

$D_1<0$

Case: $D_1=0$; solve for l.

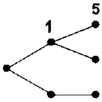
Condition: $D_2=-\frac{hf}{2}+\frac{hd}{2}-gf+gd-m-n \neq 0.$

136. Diagram: 

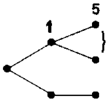
Case: $D_2 = 0$; solve for m.

Condition: $D_3 = -64p - 16d^3 + 32kd - 16f^2d + 32fd^2 - 32kf + 64g^2 + 64hg + 16h^2 \neq 0$.

Diagrams:

137. 

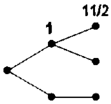
$D_3 > 0$

138. 

$D_3 < 0$

Case: $D_3 = 0$; solve for p.

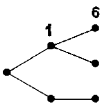
Condition: $D_4 = \frac{fdg}{2} - \frac{d^2h}{4} - \frac{3d^2g}{4} + \frac{f^2h}{4} - \frac{nd}{2} + \frac{nf}{2} + \frac{f^2g}{4} + \frac{kh}{2} + kg + \frac{ad^3}{8} - \frac{af^3}{8} - q - \frac{3afd^2}{8} + \frac{3af^2d}{8} \neq 0$.

139. Diagram: 

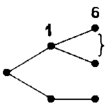
Case: $D_4 = 0$; solve for q.

Condition: $D_5 \neq 0$ (see [9]).

Diagrams:

140. 

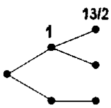
$D_5 > 0$

141. 

$D_5 < 0$

Case: $D_5 = 0$; solve for n.

Condition: $D_6 \neq 0$ (see [9]).

142. Diagram: 

Case: $D_6 = 0$; solve for a.

The denominator of a (see [9]) factors as $(2g+h)^3(f-d)$.

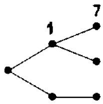
Let $f = d$.

Under the condition $D_7 = h^4 + 12h^2g^2 + 8hg^3 + 6h^3g - 8kg^2d - 8hgkd - 2h^2kd - 2k^3 \neq 0$, the resulting family has the same diagram.

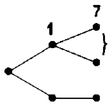
In the case $D_7 = 0$; solve for d.

Condition: $D_8 \neq 0$ (see [9]).

Diagrams:

143. 

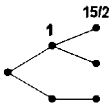
$D_8 > 0$

144. 

$D_8 < 0$

Case: $D_8 = 0$; solve for a.

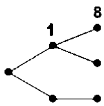
Condition: $D_9D_{10} = (32g^4 - 12k^3 + 40hg^3 + 12h^2g^2 - 2h^3g - h^4)(4k^3 + h^4 + 6h^3g + 12h^2g^2 + 8hg^3) \neq 0$.

145. Diagram: 

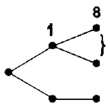
Case: $D_9 = 0$; solve for k.

Condition: $D_{11} = (52g-h)(2g+h)^4(4g-h) \neq 0$.

Diagrams:

146. 

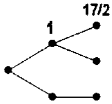
$D_1 1 > 0$

147. 

$D_1 1 < 0$

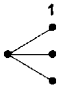
Case: $h = 52g$.

Condition: $g \neq 0$.

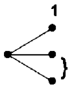
148. Diagram: 

The latter diagram represents a D_{19} singularity, and by Gudkov's theorem a D_{20} is not possible for sixth degree curves.

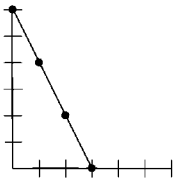
Tangent cone: $(y-x)(y-2x)(y-3x)$.

149. Diagram: 

Tangent cone: $(y-x)(y^2+x^2)$.

150. Diagram: 

Tangent cone: y^3.

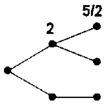
Newton polygon: 

We wish to consider the general family $(y-nx^2)^2(y-px^2) + ax^5y + bx^3y^2 + cxy^3 + dx^4y^2 + ex^2y^3 + fy^4 + gx^3y^3 + hxy^4 + jx^2y^4 + ky^5 + lxy^5 + my^6 = 0$.

It turns out that Maple computation with this family is too big for our computer. The singular points at the origin for the curves in this family belong to the J_k series [1]. By the theorem of Gudkov [4, p. 2], the maximum possible k for an irreducible sextic curve is 19. Consider the above family with $n = 1$, $p = 2$. It suffices to show that, by Maple computation, this family has all J_k singular points for $11 \leq k \leq 19$.

Quasihomogeneous factors: $(y-x^2)^2(y-2x^2)$.

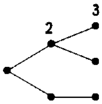
Condition: $a+b+c \neq 0$.

151. Diagram: 

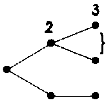
Case: $a = -b - c$.

Condition: $D_1 = 4c^2 + 4cb + b^2 + 4d + 4f + 4e \neq 0$.

Diagrams:

152. 

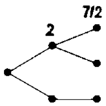
$D_1 > 0$

153. 

$D_1 < 0$

Case: $D_1 = 0$; solve for e.

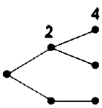
Condition: $D_2 = g + h + c^3 + c^2 b + \frac{cb^2}{4} - dc - \frac{db}{2} + fc + \frac{fb}{2} \neq 0$.

154. Diagram: 

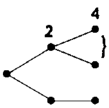
Case: $D_2 = 0$; solve for g.

Condition: $D_3 \neq 0$ (not shown here; see [9]).

Diagrams:

155. 

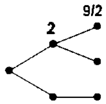
$D_3 > 0$

156. 

$D_3 < 0$

Case: $D_3 = 0$; solve for j.

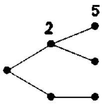
Condition: $D_4 \neq 0$ (not shown here; see [9]).

157. Diagram: 

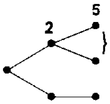
Case: $D_4 = 0$; solve for l.

Condition: $D_5 \neq 0$ (not shown here; see [9]).

Diagrams:

158. 

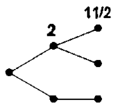
$D_5 > 0$

159. 

$D_5 < 0$

Case: $D_5 = 0$; solve for m.

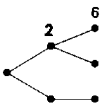
Condition: $D_6D_7 = (-72c^2b - 26cb^2 + 16dc + 6db - 24fc - 10fb - 4h - 64c^3 - 3b^3) \times D_7 \neq 0$.

160. Diagram: 

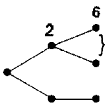
Case: $D_7 = 0$; solve for k.

Condition: $-256D_8{D_9}^2 = -256(-160c^2 - 21b^2 + 24d - 40f - 120cb){D_9}^2 \neq 0$.

Diagrams:

161. 

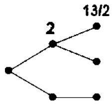
$D_8 < 0$

162. 

$D_8 > 0$

Case: $D_8 = 0$; solve for d.

Condition: $D_{10} = 32fc + 512c^3 + 576c^2b + 216cb^2 - 48h + 27b^3 \neq 0$.

163. Diagram: 

Groebner basis techniques show that the resulting family contains irreducible curves.

Case: $D_{10} = 0$; solve for h.

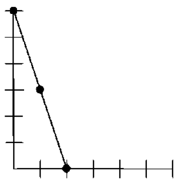
Then each curve in the resulting family is reducible (see [9]!).

Multiplicity 2

The tangent cone $y^2 - x^2$ gives an A_1 singular point, while the tangent cone $y^2 + x^2$ gives an A_1^* singular point.

Tangent cone: y^2.

The singular points A_2, A_3, A_3^*, A_4, A_5, and A_5^* are all easily obtained because they can be generated from Newton polygons whose segments do not have multiple quasihomogeneous factors.

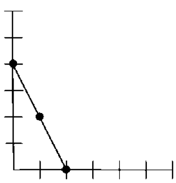
Consider the Newton polygon: 

$(y + x^3)^2 + ax^5y + bx^2y^2 + cx^4y^2 + dxy^3 + ex^3y^3 + fy^4 + gx^2y^4 + hxy^5 + gy^6 = 0.$

Maple computation with this family (see [9]) gives the following singular points:

A_k, where $6 \leq k \leq 17$, $k \neq 12, 16$, and A_k^*, where $k = 7, 9, 11, 13, 15, 17$.

Note here that $A_{12}, A_{16}, A_{18}, A_{19}$ are not obtained.

Consider the Newton polygon: 

We would like to consider the family $(y - x^2)^2 + ax^5 + bx^3y + cxy^2 + dx^6 + ex^4y + fx^2y^2 + gy^3 + hx^5y + jx^3y^2 + kxy^3 + lx^4y^2 + mx^2y^3 + ny^4 + px^3y^3 + qxy^4 + rx^2y^4 + sy^5 + txy^5 + uy^6 = 0$. However, Maple computation with this family is too big for our computers. Gudkov's theorem [4, p. 2] establishes a bound of $k = 20$ for an A_k singularity of an irreducible sextic curve.

However, in fact it is known [10] that the maximum such k is 19.

It is interesting to note that Maple computation [9] with each of the families

$(y+x^2)^2+(y+x^2)^3+hx^5y+jx^3y^2+kxy^3+lx^4y^2+mx^2y^3+ny^4+tx^3y^3+uxy^4+px^2y^4+qy^5+rxy^5+sy^6=0$ and
$(y+x^2)^2+dx^6+ex^4y+fx^2y^2+gy^3+hx^5y+jx^3y^2+kxy^3+y^2(y+x^2)^2+xy^3(y+x^2)+px^2y^4+qy^5+rxy^5+sy^6=0$ yields an A_{16} singular point (also an A_{12}).

Note also that Maple computation [9] with

$$(y+x^2)^2+dx^6+ex^4y+fx^2y^2+gy^3+hx^5y+jx^3y^2+kxy^3+lx^4y^2+mx^2y^3+ny^4+px^3y^3+qxy^4+rx^2y^4+sy^5+txy^5+uy^6=0$$

gives an A_{18} singular point.

By referring to [2], we know that the family

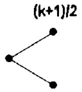
$$(y-x^2)^2+ax(y-x^2)^2+(y-x^2)^2(by+cx^2)+xy(y-x^2)(dy+ex^2)+y^2(y-x^2)(fy+gx^2)+hxy^3(y-x^2)+y^4(jy+kx^2)+lxy^5+my^6=0$$

contains a curve with an A_{19} singular point. However, it is interesting to note that this singular point cannot be obtained by our usual Maple computations [9].

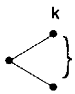
To summarize, the list of singular points of multiplicity two for real irreducible sextic curves is the following:

A_k, $1\le k\le 19$ and A^*_{2k+1}, $0\le k\le 9$.

Diagrams:

164.–182.


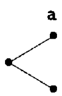
$1\le k\le 19$

183.–191.


$1\le k\le 10$

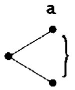
3 Summary of Singular Points of Irreducible Real Sextic Curves

Multiplicity 2



$a=1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4,\frac{9}{2},5,\frac{11}{2},6,\frac{13}{2},7,\frac{15}{2},8,\frac{17}{2},9,\frac{19}{2},10$

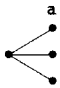
$A_1, A_2, A_3, A_4, A_5, \ldots, A_{19}$



$$a = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

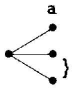
$$A_1^*, A_3^*, A_5^*, A_7^*, A_9^*, A_{11}^*, A_{13}^*, A_{15}^*, A_{17}^*, A_{19}^*$$

Multiplicity 3



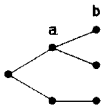
$$a = 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4$$

$$D_4, E_7, J_{10}, E_{13}, J_{3,0}, E_{19}$$



$$a = 1, \tfrac{4}{3}, \tfrac{5}{3}, 2, \tfrac{7}{3}, \tfrac{8}{3}, 3, \tfrac{10}{3}, \tfrac{11}{3}$$

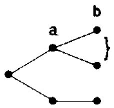
$$D_4^*, E_6, E_8, J_{10}^*, E_{12}, E_{14}, J_{3,0}^*, E_{18}, E_{20}$$



$a = 1$; $b = \tfrac{3}{2}, \tfrac{4}{2}, \tfrac{5}{2}, \dots, \tfrac{17}{2}$ or,
$D_5, D_6, D_7, \dots, D_{19}$

$a = 2$; $b = \tfrac{5}{2}, \tfrac{6}{2}, \tfrac{7}{2}, \tfrac{8}{2}, \tfrac{9}{2}, \tfrac{10}{2}, \tfrac{11}{2}, \tfrac{12}{2}, \tfrac{13}{2}$ or,
$J_{11}, J_{12}, J_{13}, J_{14}, J_{15}, J_{16}, J_{17}, J_{18}, J_{19}$

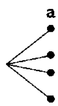
$a = 3$; $b = \tfrac{7}{2}$
$J_{3,1}$



$a = 1$; $b = 2, 3, 4, 5, 6, 7, 8$ or,
$D_6^*, D_8^*, \dots, D_{18}^*$

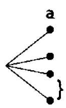
$a = 2$; $b = 3, 4, 5, 6$
$J_{12}^*, J_{14}^*, J_{16}^*, J_{18}^*$

Multiplicity 4



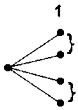
$$a = 1, \tfrac{3}{2}$$

$$X_9, W_{1,0}$$

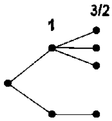


$$a = 1, \tfrac{4}{3}, \tfrac{5}{4}$$

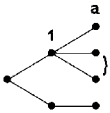
$$X_9^*, W_{13}, W_{12}$$



X_9^{**}

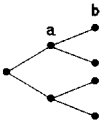


Z_{12}



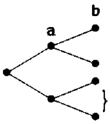
$a = \frac{4}{3}, \frac{5}{3}$

Z_{11}, Z_{13}



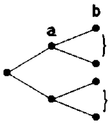
$$(a, b) = (\tfrac{3}{2}, 2), (\tfrac{3}{2}, \tfrac{5}{2}), (1, \tfrac{3}{2}), (1, 2), (1, \tfrac{5}{2}), (1, 3), (1, \tfrac{7}{2})$$

$$W_{1,2}, W_{1,4}, Y^1_{1,1}, Y^1_{2,2}, Y^1_{3,3}, Y^1_{4,4}, Y^1_{5,5}$$



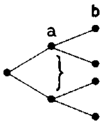
$$(a, b) = (\tfrac{3}{2}, \tfrac{7}{4}), (\tfrac{3}{2}, \tfrac{9}{4}), (\tfrac{3}{2}, \tfrac{11}{4}), (1, 2), (1, 3)$$

$$W_{1,1}, W_{1,3}, W_{1,5}, Y^{1*2}_{2,2}, Y^{1*3}_{4,4}$$



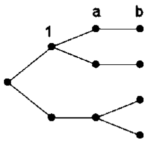
$$(a, b) = (\tfrac{3}{2}, 2), (\tfrac{3}{2}, \tfrac{5}{2}), (1, 2), (1, 3)$$

$$W^*_{1,2}, W^*_{1,4}, Y^{1*2*2}_{2,2}, Y^{1*3*3}_{4,4}$$

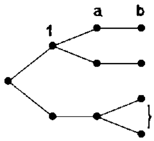


$$(a, b) = (1, \tfrac{3}{2}), (1, 2), (1, \tfrac{5}{2}), (1, 3), (1, \tfrac{7}{2})$$

$$Y^{1*1}_{1,1}, Y^{1*1}_{2,2}, Y^{1*1}_{3,3}, Y^{1*1}_{4,4}, Y^{1*1}_{5,5}$$

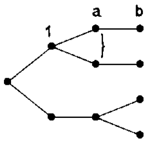


$$(a, b) = (\tfrac{3}{2}, 2), (\tfrac{3}{2}, \tfrac{5}{2}), (\tfrac{3}{2}, 3), (\tfrac{3}{2}, \tfrac{7}{2}), (\tfrac{3}{2}, 4), (\tfrac{3}{2}, \tfrac{9}{2}),$$
$$(\tfrac{3}{2}, 5), (\tfrac{3}{2}, \tfrac{11}{2}), (2, \tfrac{5}{2}), (2, 3), (2, \tfrac{7}{2}), (2, 4), (2, \tfrac{9}{2}),$$
$$(2, 5), (\tfrac{5}{2}, 3), (\tfrac{5}{2}, \tfrac{7}{2}), (\tfrac{5}{2}, 4), (\tfrac{5}{2}, \tfrac{9}{2}), (3, \tfrac{7}{2}), (3, 4)$$
$$Y^1_{1,2}, \ldots, Y^1_{1,9}, Y^1_{2,3}, \ldots, Y^1_{2,8}, Y^1_{3,4}, \ldots, Y^1_{3,7}, Y^1_{4,5}, Y^1_{4,6}$$



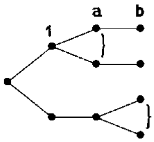
$$(a,b) = (\tfrac{3}{2},2), (\tfrac{3}{2},3), (\tfrac{3}{2},4), (\tfrac{3}{2},5), (2,3), (2,4), (2,5), (\tfrac{5}{2},3), (\tfrac{5}{2},4), (3,4)$$

$$Y_{1,2}^{1*2}, Y_{1,4}^{1*3}, Y_{1,6}^{1*4}, Y_{1,8}^{1*5}, Y_{2,4}^{1*3}, Y_{2,6}^{1*4}, Y_{2,8}^{1*5}, Y_{3,4}^{1*3}, Y_{3,6}^{1*4}, Y_{4,6}^{1*4}$$



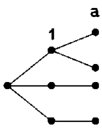
$$(a,b) = (2,\tfrac{5}{2}), (2,3), (2,\tfrac{7}{2}), (2,4), (2,\tfrac{9}{2}), (2,5), (3,\tfrac{7}{2}), (3,4)$$

$$Y_{2,3}^{1*2}, Y_{2,4}^{1*2}, Y_{2,5}^{1*2}, Y_{2,6}^{1*2}, Y_{2,7}^{1*2}, Y_{2,8}^{1*2}, Y_{4,5}^{1*3}, Y_{4,6}^{1*3}$$



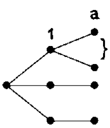
$$(a,b) = (2,3), (2,4), (2,5), (3,4)$$

$$Y_{2,4}^{1*2*3}, Y_{2,6}^{1*2*4}, Y_{2,8}^{1*2*5}, Y_{4,6}^{1*3*5}$$



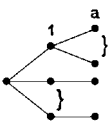
$$a = \tfrac{3}{2}, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4, \tfrac{9}{2}, 5, \tfrac{11}{2}$$

$$X_{1,1}, X_{1,2}, X_{1,3}, X_{1,4}, X_{1,5}, X_{1,6}, X_{1,7}, X_{1,8}, X_{1,9}$$



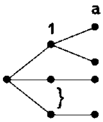
$$a = 2, 3, 4, 5$$

$$X_{1,2}^{*2}, X_{1,4}^{*3}, X_{1,6}^{*4}, X_{1,8}^{*5}$$



$$a = 2, 3, 4, 5, 6$$

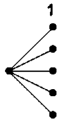
$$X_{1,2}^{*1*2}, X_{1,4}^{*1*3}, X_{1,6}^{*1*4}, X_{1,8}^{*1*5}, X_{1,10}^{*1*6}$$



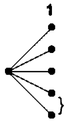
$$a = \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4, \tfrac{9}{2}, 5, \tfrac{11}{2}$$

$$X_{1,1}^{*1}, X_{1,2}^{*1}, \ldots, X_{1,9}^{*1}$$

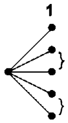
Multiplicity 5



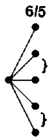
N_{16}



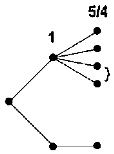
N_{16}^{*}

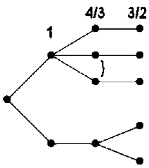


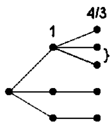
N_{16}^{**}

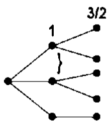


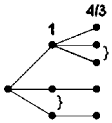
$T_{5,6}$ Under the projection $(x, y, z) \mapsto (x, y, 0)$

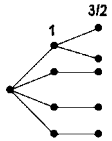


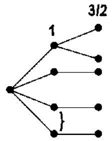












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References

1. Arnol'd, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of Differentiable Maps, vol. I. Birkhauser, Boston (1985)
2. Artal Bartolo, E., Ruber, J.C., Cogolludo Agustin, J.I.: On sextic curves with big Milnor number. In: Trends in Mathematics: Trends in Singularities, pp. 1–29. Birkhauser, Basel (2002)
3. Brieskorn, E., Knorrer, H.: Plane Algebraic Curves. Birkhauser, Basel (1986)
4. Gudkov, D.A.: Nine papers on Hilbert's 16th problem by D.A. Gudkov and G.A. Utkin. American Mathematical Society Translations, Series 2, vol. 112. American Mathematical Society, Providence (1978)
5. Milnor, J.: Singular Points of Complex Hypersurfaces. Annals of Mathematics Studies, vol. 61. Princeton University Press, Princeton (1968)
6. Urabe, T.: Combinations of rational singularities on plane sextic curves with the sum of Milnor numbers less than sixteen. In: Singularities, Banach Center Publications, vol. 20, pp. 429–456. PWN-Polish Scientific Publishers, Warsaw (1988)
7. Urabe, T.: Dynkin graphs and combinations of singularities on plane sextic curves. In: Randell, R. (ed.) Singularities, Proc., Univ. of Iowa 1986. Contemporary Math., vol. 90, pp. 295–316. American Mathematical Society, Providence (1989)
8. Walker, R.J.: Algebraic Curves. Princeton University Press, New Jersey (1950). 2nd edn. published by Springer, New York (1978)
9. Weinberg, D.A., Willis, N.J.: Technical reports. http://www.math.ttu.edu/~weinberg/maple_files.htm and http://www.whitworth.edu/academic/department/mathcomputerscience/faculty/willisnicholas/maple_files.htm
10. Yang, J.-G.: Sextic curves with simple singularities. Tohoku Math. J. **48**, 203–227 (1996)