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# Scattering by a Narrow Gap in an Impedance Plane

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## **SCATTERING BY A NARROW GAP IN AN IMPEDANCE PLANE**

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## **ABSTRACT**

For a plane wave incident on a cavity-backed gap in an impedance plane, the coupled integral equations for the induced currents have been solved numerically and the far field scattering computed. The results are compared with a quasi-analytical solution previously derived and modified for the impedance plane. For narrow gaps of widths less than  $0.15\lambda$ , the agreement is within 12 percent for H-polarization and 14 percent for E-polarization for the cavity geometries considered, limited to small surface impedances of the plane. Excellent agreement is obtained when the material filling of the gap is lossy.

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## 1. INTRODUCTION

A case of interest in radar cross section studies is the scattering from gaps and cracks in planar surfaces. Results were recently derived [1,2] for the narrow gap in a perfectly conducting ground plane. Of equal concern is the scattering from a gap of similar geometry but in a ground plane coated with a material having arbitrary dielectric properties.

The development of the solution in this paper is for planar surfaces large in extent compared to the width of the gap. As in [2], a set of coupled integral equations for the electric and magnetic currents which exist on the walls of the gap cavity and in aperture of the gap are developed by employing the equivalence principle [3]. Since a thinly coated conducting surface can be described by the impedance boundary condition [4], the half-space Green's function for an impedance plane has been developed and is applied. For plane wave incidence the integral equations are derived for a cavity of arbitrary shape filled with a homogeneous material. The equations are solved by the moment method and data for several cavities are presented.

A quasi-analytical solution was derived in [5] for the far field scattering from a uniform resistive or impedance insert in a perfectly conducting plane. In [2], the solution was applied to a narrow gap in the conducting ground plane. Accurate results were obtained by defining the surface impedance of the gap as the input impedance of the gap modeled as a shorted transmission line. Modifications are made to the quasi-analytical solution of [5] such that it can predict the scattering from a gap in an impedance plane with a small surface impedance.

The modified quasi-analytical solution is applied to the gap in the impedance plane for gap widths which are electrically small, and the results are

compared with those obtained using the coupled integral equations. The limitations are determined for which this quasi-analytical solution provides an accurate design tool.

## 2. THEORETICAL DEVELOPMENT OF THE INTEGRAL EQUATIONS

The gap geometry under consideration is the two-dimensional one shown in Fig. 1. The plane  $y = 0$  is an impedance plane of infinite extent for  $|x| > w/2$  with a surface impedance  $\bar{\eta}$ . The plane  $y = 0$  for  $|x| < w/2$  contains the aperture  $A$  of the gap cavity whose walls  $S$  are perfectly conducting. The cavity is filled with a homogeneous dielectric material of permittivity  $\epsilon_1 = \epsilon_r \epsilon$  and permeability  $\mu_1 = \mu_r \mu$ , where the quantities without subscripts refer to free space. The incident plane wave is

$$\vec{H}^i, \vec{E}^i = \hat{z} e^{-ik(x \cos \phi_0 + y \sin \phi_0)} \quad (1)$$

for H- and E-polarizations respectively, where  $k$  is the propagation constant in the free space region above the surface. A time factor  $e^{-i\omega t}$  is assumed and suppressed.

### 2.1 H-Polarization

The equivalence principle is applied to the gap for the region  $y > 0$  by shorting the gap with a perfect electric conductor in  $A$  and placing a magnetic current  $\vec{J}^* = -\hat{y} \times \vec{E}(x, 0)$  over it. The plane  $y = 0$  is then an impenetrable surface with mixed boundary conditions: that of a perfectly conducting surface for  $|x| < w/2$  and that of an impedance surface for  $|x| > w/2$ . For the tangential magnetic field, the impedance boundary condition is given by [4]

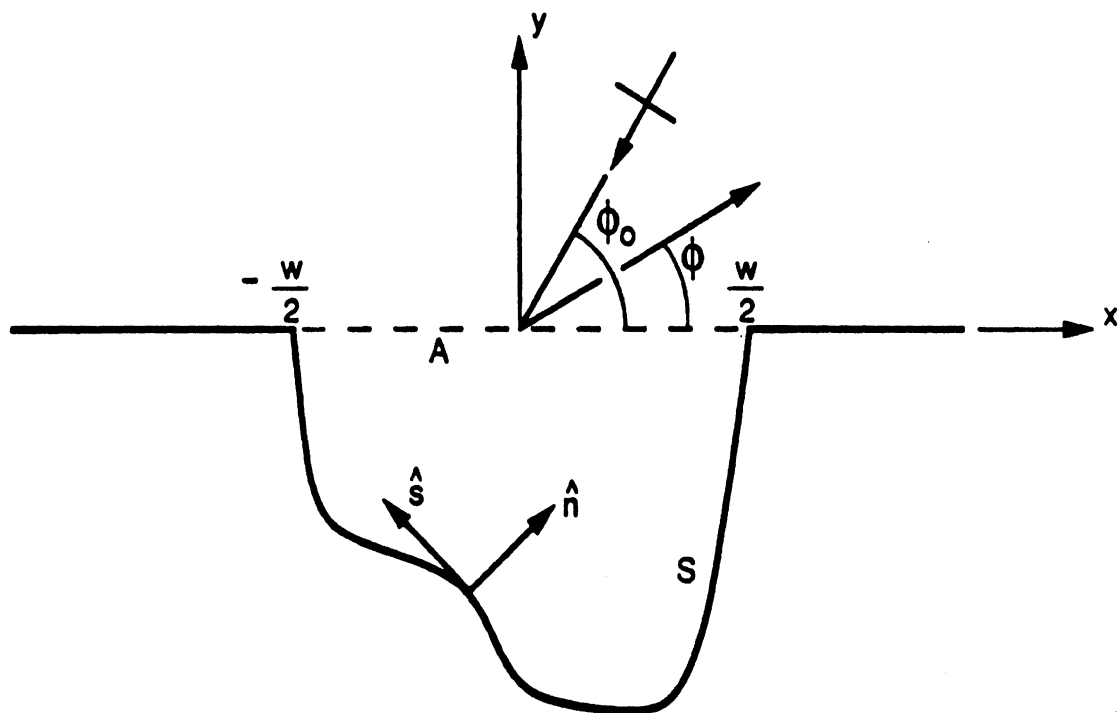


Figure 1. Narrow gap of arbitrary shape in an infinite ground plane.

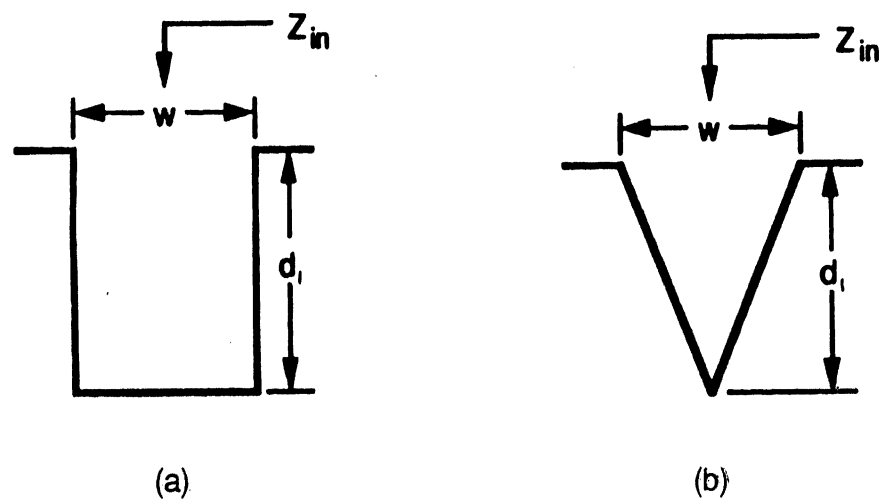


Figure 2. Rectangular and triangular gap geometries and dimensions.

$$\left( \frac{\partial}{\partial y} + \frac{ik\bar{\eta}}{Z} \right) H_z = 0 \quad (2)$$

where  $Z$  is the free space impedance. By applying Green's theorem, an integral equation for the total magnetic field can be written as

$$H_z(x,y) = H_z^i(x,y) + \int_A \left( J_x(x') \frac{\partial}{\partial y'} G(x,y;x',y') + ikY J_z^*(x') G(x,y;x',0) \right) dx' \\ + \left[ \int_{-\infty}^{-w/2} + \int_{w/2}^{\infty} \right] \left( H_z(x',0) \frac{\partial}{\partial y'} G(x,y;x',y') - G(x,y;x',0) \frac{\partial}{\partial y'} H_z(x',y') \right) dx' \quad y' \rightarrow 0$$

where  $G$  is the appropriate free space Green's function. This is certainly not a desirable integral equation to solve with the path of infinite extent over  $|x| > w/2$ . However, this path of integration over the  $y = 0$  plane can be eliminated by applying the half space Green's function for an impedance plane, which also satisfies (2).

The half space Green's function for the impedance plane is

$$G = \frac{i}{4} \left\{ H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y-y')^2} \right) - \frac{1}{\pi} \int_C \Gamma_H(\alpha) e^{ik[(x-x')\cos\alpha + |y+y'|\sin\alpha]} d\alpha \right\} \quad (3)$$

where  $H_0^{(1)}$  is the zeroth order Hankel function of the first kind and

$$\Gamma_H(\alpha) = \frac{\bar{\eta}Y - \sin\alpha}{\bar{\eta}Y + \sin\alpha}$$

is the reflection coefficient for an H-polarized plane wave incident on an impedance boundary at the angle  $\alpha$ . The path  $C$  is in the complex  $\alpha$  plane, defined in Fig. 3. Note that as  $\bar{\eta}$  approaches zero, (3) becomes the Green's function for a perfectly conducting plane for the H-polarization case. For the



numerical solution of (3), a more quickly converging integral gives

$$G = \frac{i}{4} \left\{ H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y-y')^2} \right) + H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y+y')^2} \right) - 2 \int_0^\infty \tau_H e^{-\tau_H v} H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y+y'+iv)^2} \right) dv \right\} \quad (4)$$

where

$$\tau_H = \frac{k\bar{\eta}}{Z}.$$

Equation (4) was derived using a transform technique presented in [6].

For such a Green's function to be applied, the continuity of the impedance boundary condition over the entire  $y = 0$  plane is necessary. This is accomplished by impressing a magnetic current source equivalent to  $\bar{\eta}H_z$  over

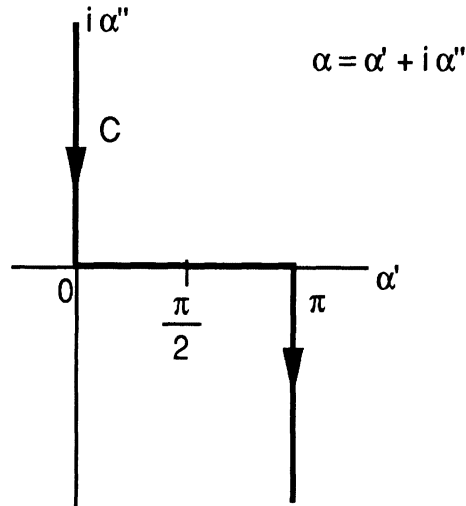


Figure 3. Path of integration in the complex  $\alpha$  plane.

A. Applying then the half space Green's function of (4),

$$H_z(x,y) = H_z^i(x,y) - H_z^r(x,y) + ikY \int_A J_z^*(x') G(x,y;x',0) dx' - \frac{ik}{Z} \int_A \bar{\eta} H_z(x',0) G(x,y;x',0) dx' \quad (5)$$

where

$$H_z^r(x,y) = \frac{\bar{\eta}Y - \sin\phi_0}{\bar{\eta}Y + \sin\phi_0} e^{-ik(x \cos\phi_0 - y \sin\phi_0)}$$

is the reflected plane wave, and  $Y = 1/Z$ . The second integral in (5) is the correction term for the additional impressed current, ensuring the original boundary condition for the total field at the shorted gap. That is, it removes the contribution of the scattered field due to the surface  $\bar{\eta}$  over  $A$  which was not part of the original system. Observing the field in the aperture,

$$H_z(x,0) = \frac{2 \sin\phi_0}{\bar{\eta}Y + \sin\phi_0} e^{-ikx \cos\phi_0} + ikY \int_A [J_z^*(x') - \bar{\eta}J_s(x')] G(x,0;x',0) dx' \quad (6)$$

where  $J_s(x') = [\hat{y} \times \bar{H}(x',0)] \cdot \hat{s}$  from (5). As shown in Fig. 1,  $\hat{s}$  is tangential to the surface of the cavity wall, and in the aperture,  $\hat{s} = \hat{x}$ .

For the region  $y < 0$  occupied by the cavity, a magnetic current  $-\bar{J}^*$  is placed just below  $A$  to ensure the continuity of the tangential electric field in the open gap. The expression for the magnetic field in the shorted cavity is the same as that constructed in [2] since the region is independent of the impedance boundary. The integral equation for the currents on the cavity walls is [2,Eq. (7)],

$$J_s(s) = \frac{kY}{2} \epsilon_r \int_A J_z^*(x') H_0^{(1)} \left( k_1 \sqrt{(x-x')^2 + y^2} \right) dx' + \frac{ik_1}{2} \int_{S+A} J_s(s') \sin \beta' H_1^{(1)} \left( k_1 \sqrt{(x-x')^2 + (y-y')^2} \right) ds' \quad (7)$$

where  $k_1 = k \sqrt{\epsilon_r \mu_r}$  and

$$\sin \beta' = \hat{z} \cdot \frac{(x-x') \hat{x} + (y-y') \hat{y}}{\sqrt{(x-x')^2 + (y-y')^2}} \times \hat{s}', \quad (8)$$

valid at all points of S and A. To ensure the continuity of  $H_z$  through the aperture, the field expression for (7) [2, Eq. (9)] is matched to (6), resulting in

$$J_s(x) = \frac{2 \sin \phi_0}{\bar{\eta} Y + \sin \phi_0} e^{-ikx \cos \phi_0} + ikY \int_A [J_z^*(x') - \bar{\eta} J_s(x')] G(x, 0; x', 0) dx' \quad (9)$$

valid for  $x$  in A, and (7) and (9) constitute a pair of coupled integral equations for  $J_z^*$  and  $J_s$ .

In the far field, the large argument expression for the Green's function of (3) is found by saddle point integration, allowing C to become the path of steepest descent about  $\alpha = \pi/2$ . The scattered magnetic field is then given as

$$H_z^s = \sqrt{\frac{2}{\pi k \rho}} e^{i(k\rho - \pi/4)} P_H(\phi, \phi_0) \quad (10)$$

where

$$P_H(\phi, \phi_0) = -\frac{kY}{2} \frac{1}{\bar{\eta} Y + 1} \int_A [J_z^*(x') - \bar{\eta} J_s(x')] e^{-ikx' \cos \phi} dx' \quad (11)$$

Thus the far field amplitude is given for the currents  $J_z^*$  and  $J_s$  over A determined from (7) and (9).

## 2.2 E-Polarization

The impedance boundary condition for the total tangential electric field is

[4]

$$\left( \frac{\partial}{\partial y} + i \frac{kZ}{\bar{\eta}} \right) E_z = 0 \quad (12)$$

The half-space Green's function for an impedance plane with an E-polarized incident plane wave satisfying the same boundary condition is

$$G = \frac{i}{4} \left\{ H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y-y')^2} \right) + \frac{1}{\pi} \int_C \Gamma_E(\alpha) e^{ik[(x-x')\cos\alpha + |y+y'|\sin\alpha]} d\alpha \right\} \quad (13)$$

where

$$\Gamma_E(\alpha) = \frac{\bar{\eta}Y - \csc\alpha}{\bar{\eta}Y + \csc\alpha}.$$

Note again that as  $\bar{\eta}$  approaches zero, (13) becomes the Green's function for a perfectly conducting plane under E-polarized illumination. For a quickly convergent integral, the same expression is used as that given in (4), with  $\tau_H$  replaced with

$$\tau_E = \frac{kZ}{\bar{\eta}}.$$

Using the same approach as for the H-polarization case, the total field for the region  $y > 0$  is

$$\begin{aligned} E_z(x,y) = & E_z^i(x,y) + E_z^r(x,y) - \int_A J_x^*(x') \frac{\partial}{\partial y'} G(x,y;x',y') \Big|_{y' \rightarrow 0} dx' \\ & - \int_A E_z(x',0) \frac{\partial}{\partial y'} G(x,y;x',y') \Big|_{y' \rightarrow 0} dx' \end{aligned} \quad (14)$$

where  $J_x^*$  is the assumed equivalent magnetic current on A and

$$E_z^r(x,y) = \frac{\bar{\eta}Y - \csc\phi_0}{\bar{\eta}Y + \csc\phi_0} e^{-ik(x \cos\phi_0 - y \sin\phi_0)}$$

is the reflected plane wave. The second integral in (14) accounts for the scattering due to the added impressed current equivalent to  $E_z(x',0)$  for the impedance plane. The dependence of this current on the surface impedance is shown more explicitly using [4] the boundary condition  $E_z = -\bar{\eta}H_x$ .

From  $H_x = -\frac{iY}{k} \frac{\partial E_z}{\partial y}$ , the tangential component of the magnetic field in the aperture is

$$H_x(x,0) = -\frac{2Y}{\bar{\eta}Y + \csc\phi_0} e^{-ikx \cos\phi_0} + \frac{iY}{k} \lim_{y \rightarrow 0} \frac{\partial}{\partial y} \int_A [J_x^*(x') + \bar{\eta}J_z(x')] \frac{\partial}{\partial y'} G(x,y;x',y') \Big|_{y' \rightarrow 0} dx' \quad (15)$$

In the region  $y < 0$  occupied by the cavity, the field source is the equivalent magnetic current  $\hat{J}_x^*$  in the aperture and the contribution from the electric current  $J_z$  on the walls of the cavity. The integral equation constructed for the currents is [2,Eq. (15)]

$$J_z(s) = \frac{Y}{2k\mu_r} (\hat{n} \cdot \nabla) \frac{\partial}{\partial y} \int_A J_x^*(x') H_0^{(1)} \left( k_1 \sqrt{(x-x')^2 + y^2} \right) dx' + \frac{ik_1}{2} \int_{S+A} J_z(s') \sin\beta H_1^{(1)} \left( k_1 \sqrt{(x-x')^2 + (y-y')^2} \right) ds' \quad (16)$$

where

$$\sin\beta = \hat{z} \cdot \frac{(x-x') \hat{x} + (y-y') \hat{y}}{\sqrt{(x-x')^2 + (y-y')^2}} \times \hat{s} \quad (17)$$

valid at all points of S and A. When (15) is matched to the field expression for (16) with the observation point in the aperture,

$$J_z(x) = -\frac{2Y}{\bar{\eta}Y + \csc\phi_0} e^{-ikx \cos\phi_0} + \frac{iY}{k} \lim_{y \rightarrow 0} \frac{\partial}{\partial y} \int_A \left[ J_x^*(x') + \bar{\eta}J_z(x') \right] \frac{\partial}{\partial y'} G(x,y;x',y') \Big|_{y' \rightarrow 0} dx' \quad (18)$$

for  $x$  in  $A$ . Equations (16) and (18) are a pair of coupled integral equations for the currents  $J_x^*$  and  $J_z$ .

A third integral equation was developed in [2] for the electric field in the cavity of the gap. When the boundary condition on the perfectly conducting surface was applied, the expression for the currents on  $A$  and  $S$  came to be

$$J_x^*(x) = \frac{i}{2} \int_A J_x^*(x') \frac{\partial}{\partial y} H_0^{(1)} \left( k_1 \sqrt{(x-x')^2 + y^2} \right) dx' + \frac{kZ\mu_r}{2} \int_{S+A} J_z(s') H_0^{(1)} \left( k_1 \sqrt{(x-x')^2 + (y-y')^2} \right) ds' \quad (19)$$

where  $J_x^*(x)$  is non-zero only in  $A$ . Equations (18) and (19) are the pair of integral equations used to compute  $J_x^*$  and  $J_z$ . The scattering in the far field for E-polarization has a similar expression as that in (10), giving the far field amplitude

$$P_E(\phi, \phi_0) = -\frac{k}{2} \frac{\sin\phi}{\bar{\eta}Y + 1} \int_A \left[ J_x^*(x') + \bar{\eta}J_z(x') \right] e^{-ikx' \cos\phi} dx'. \quad (20)$$

### 3. APPLICATION OF THE QUASI-ANALYTICAL SOLUTION

Consider an impedance insert of width  $w$  and characterized by  $\eta$  in an impedance plane  $\bar{\eta}$ . For the H-polarization case, as in Section 2.1, the application of Green's theorem for the region  $y > 0$  gives

$$H_z(x,y) = H_z^i(x,y) - H_z^r(x,y) + \frac{ik}{Z} \int_{-w/2}^{w/2} (\eta - \bar{\eta}) H_z(x',0) G(x,y;x',0) dx' \quad (21)$$

where the incident and reflected fields are those described in (5) and  $G$  is the Green's function of (4), satisfying (2) in the plane  $y = 0$ . The integral of (21) is the scattering due to the impedance insert, and this clearly vanishes for  $\eta = \bar{\eta}$ .

The Green's function of (4) can be written in the form, for  $y' = 0$ ,

$$G(x,y;x',0) = \frac{i}{2} H_o^{(1)} \left( k \sqrt{(x-x')^2 + y^2} \right) g(x,y;x',0|\bar{\eta}) \quad (22)$$

where the normalized half space impedance Green's function is

$$g(x,y;x',0|\bar{\eta}) = \left[ 1 - \frac{L_H(x,y;x',0|\bar{\eta})}{H_o^{(1)} \left( k \sqrt{(x-x')^2 + y^2} \right)} \right]$$

for which the function  $L_H$  is the integral of (4). We note that as  $\bar{\eta}$  approaches zero,  $L_H$  approaches zero, and as  $\bar{\eta}$  approaches infinity,  $L_H$  approaches  $H_o^{(1)}$ .

Thus  $|g|$  varies from one to zero for these limits. To remove the coordinate dependence,  $g$  is averaged over  $-w/2 < x < w/2$ , for  $x'$  and  $y$  set to zero. By curve fitting the numerically generated magnitude and phase, this average is found to be approximated analytically to within 5 percent as

$$g_H(w|\bar{\eta}) = e^{-c(w) \bar{\eta}/Z} e^{i \psi(w|\bar{\eta})} \quad (23)$$

where

$$c(w) = 0.245 + 1.267w/\lambda$$

$$\psi(w|\bar{\eta}) = 1 - e^{-(0.098 + 1.760w/\lambda) \bar{\eta}/Z}$$

for  $0.02 < w/\lambda < 0.15$  and  $\bar{\eta}/Z < 2$ .

Now taking the observation point to be on the insert, (21) can be approximated by

$$H_z(x,0) = \frac{2 \sin \phi_o}{\bar{\eta} Y + \sin \phi_o} e^{-ikx \cos \phi_o} - \frac{k}{2Z} (\eta - \bar{\eta}) g_H(w|\bar{\eta}) \int_{-w/2}^{w/2} H_z(x',0) H_o^{(1)}(k|x-x'|) dx' \quad (24)$$

where the averaging of  $g$  has enabled us to remove the  $\bar{\eta}$  dependence from the integral. The assumption that  $g$  is constant over  $w$  should hold for small  $w$  and small  $\bar{\eta}$ . The integral equation (24) can be written in the form [5]

$$-\frac{i}{2} \int_{-1}^1 J_1(\zeta') H_o^{(1)}\left(\frac{kw}{2} |\zeta - \zeta'|\right) d\zeta' = e^{-i\frac{kw}{2} \zeta \cos \phi_o} + a J_1(\zeta) \quad (25)$$

where

$$J_1(\zeta) = \frac{ikw}{2} \frac{\eta - \bar{\eta}}{Z} \frac{\bar{\eta} Y + \sin \phi_o}{2 \sin \phi_o} g_H(w|\bar{\eta}) H_z(x,0)$$

and

$$a = \frac{2i}{kw} \frac{Z}{\eta - \bar{\eta}} \frac{1}{g_H(w|\bar{\eta})} . \quad (26)$$

Referring to (11) and the integral of (21), the far field amplitude written in terms of  $J_1$  is

$$P_H(\phi, \phi_o) = i \frac{1}{\bar{\eta} Y + 1} \frac{1}{g_H(w|\bar{\eta})} \frac{\sin \phi_o}{\bar{\eta} Y + \sin \phi_o} \int_{-1}^1 J_1(\zeta') e^{-i\frac{kw}{2} \zeta' \cos \phi_o} d\zeta' . \quad (27)$$

A simplification is made using [5]

$$J_1(\zeta) = \left[ 1 + \frac{iA g_H(w|\bar{\eta})}{\pi} (\bar{\eta} Y + 1) \frac{\bar{\eta} Y + \sin \phi_o}{\sin \phi_o} P_H(\phi, \phi_o) \right] J_2(\zeta)$$

where the integral equation that the modified current  $J_2$  satisfies is [2,5]



$$\frac{1}{\pi} \int_{-1}^1 J_2(\zeta') \ell n |\zeta - \zeta'| d\zeta' = 1 + a J_2(\zeta)$$

for  $-1 < \zeta < 1$  and

$$A = \ell n \frac{kw}{4} + \gamma - i \frac{\pi}{2}$$

where  $\gamma = 0.5772157....$  is Euler's constant. The far field amplitude (27)

becomes

$$P_H(\phi, \phi_o) = i\pi \frac{1}{\bar{\eta}Y + 1} \frac{1}{g_H(w|\bar{\eta})} \frac{\sin\phi_o}{\bar{\eta}Y + \sin\phi_o} \left[ A + \frac{1}{K_H(a)} \right]^{-1} \quad (28)$$

with

$$K_H(a) = \frac{1}{\pi} \int_{-1}^1 J_2(\zeta) d\zeta ,$$

for which an approximate expression is [2]

$$K_H(a) = - \frac{1}{\frac{\pi a}{2} + \ell n 2 + 0.1} \quad (29)$$

with  $a$  given in (26). Thus with the modifications to  $a$  and  $P_H$ , the same quasi-analytical expressions are used to solve for the scattering of the impedance insert in an impedance plane.

Similarly, for E-polarization, the total field for the region  $y > 0$  is

$$E_z(x, y) = E_z^i(x, y) + E_z^r(x, y) + \int_{-w/2}^{w/2} \left[ E_z(x', 0|\eta) - E_z(x', 0|\bar{\eta}) \right] \frac{\partial}{\partial y'} G(x, y; x', y') \Big|_{y' \rightarrow 0} dx' , \quad (30)$$

and the correlation of the field with the respective impedance surface is shown explicitly. The corresponding expression for the tangential magnetic field over the insert is

$$H_x(x,0) = -\frac{2Y}{\bar{\eta}Y + \csc\phi_0} e^{-ikx \cos\phi_0} - \frac{i}{kZ} \lim_{y \rightarrow 0} \frac{\partial}{\partial y} \int_{-w/2}^{w/2} (\eta - \bar{\eta}) H_x(x',0) \frac{\partial}{\partial y'} G(x,y;x',y') \Big|_{y' \rightarrow 0} dx' \quad (31)$$

where  $G$  is (13). When (13) is expressed in the form of (4), the partial derivatives in (31) render the source and image Hankel functions zero for  $y' = 0$ , and  $G$  is given by (22) with

$$g(x,y;x',0|\bar{\eta}) = \frac{L_E(x,y;x',0|\bar{\eta})}{H_o^{(1)}\left(k\sqrt{(x-x')^2 + y^2}\right)}$$

where  $L_E$  is the integral of (4) with  $\tau_H$  replaced with  $\tau_E$ . For this case we note that as  $\bar{\eta}$  approaches zero,  $L_E$  approaches  $H_o^{(1)}$ , and as  $\bar{\eta}$  approaches infinity,  $L_E$  approaches zero. Thus  $|g|$  varies from one to zero for these limits. The average over  $w$  of the normalized half space impedance Green's function is approximated analytically within 10 percent as

$$g_E(w|\bar{\eta}) = e^{-d(w)\bar{\eta}/Z} e^{i\vartheta(w|\bar{\eta})} \quad (32)$$

where

$$d(w) = 1.558 - 4.226w/\lambda$$

$$\vartheta(w|\bar{\eta}) = - (0.380 - 0.80w/\lambda) \left(1 - e^{-2.586\sqrt{\bar{\eta}/Z}}\right)$$

for  $0.025 < w/\lambda < 0.15$  and  $\bar{\eta}/Z < 2$ .

The tangential magnetic field on the insert is then approximated as

$$H_x(x,0) = -\frac{2Y}{\bar{\eta}Y + \csc\phi_0} e^{-ikx \cos\phi_0} - \frac{i}{kZ} (\eta - \bar{\eta}) g_E(w|\bar{\eta}) \left( k^2 + \frac{\partial^2}{\partial x^2} \right) \int_{-w/2}^{w/2} H_x(x',0) H_o^{(1)}(k|x-x'|) dx' . \quad (33)$$

Given this approximation, (33) can be equated to [5]

$$\left[ \frac{\partial^2}{\partial \zeta^2} + \left( \frac{kW}{2} \right)^2 \right] \frac{1}{2i} \int_{-1}^1 J_3(\zeta') H_o^{(1)}\left( \frac{kW}{2} |\zeta - \zeta'| \right) d\zeta' = e^{-i\frac{kW}{2} \zeta \cos\phi_0} - bJ_3(\zeta)$$

with

$$J_3(\zeta) = -\frac{2i}{kW} \frac{\eta - \bar{\eta}}{Z} \frac{\bar{\eta}Y + \csc\phi_0}{2Y} g_E(w|\bar{\eta}) H_x(x,0) \quad (34)$$

and

$$b = -\frac{ikw}{2} \frac{Z}{\eta - \bar{\eta}} \frac{1}{g_E(w|\bar{\eta})} . \quad (35)$$

Referring to (20) and using (32), the far field amplitude for the E-polarization case is

$$P_E(\phi, \phi_0) = -\frac{i\pi}{4} (kW)^2 \frac{\sin\phi}{\bar{\eta}Y + 1} \frac{1}{\bar{\eta}Y + \csc\phi_0} \frac{1}{g_E(w|\bar{\eta})} K_E(b) \quad (36)$$

with  $K_E$  approximated by [2,5]

$$K_E(b) = \frac{0.62}{b + 1.15} \frac{(b + 4.08)(b + 7.26)(b + 10.37)(b + 13.43)(b + 16.46)}{(b + 4.27)(b + 7.37)(b + 10.45)(b + 13.49)(b + 16.50)} . \quad (37)$$

As in [2], the analogy is drawn from the impedance insert to the narrow gap by equating the surface impedance  $\eta$  to the input impedance of the gap modeled as a transmission line. The expressions for various gap and cavity

configurations are contained in [2], and the far field amplitude  $P_H$  and  $P_E$  can then be calculated accordingly by (28) and (36), respectively.

#### 4. NUMERICAL RESULTS

The integral equation pairs (7), (9) and (18), (19) for H- and E-polarizations respectively were programmed for solution by the method of moments, using pulse basis and point matching functions. The half space impedance Green's function of (4) and its derivatives were evaluated analytically in handling the singularities of the integration and numerically otherwise as described in Appendix A. The application of the method of moments is described in Appendix B, which also contains the program listing used for generating the results. The quasi-analytical expressions of Section 3 were programmed for solution, as listed in Appendix C. The upper limit of  $w/\lambda = 0.15$  was determined for the applicability of the quasi-analytical solution in [2], and this limit is maintained for the results listed here.

In Figs. 4 and 5 the magnitude and phase of the far field amplitude  $P_H(\pi/2, \pi/2)$  are shown as a function of depth for a rectangular air-filled gap of width  $w/\lambda = 0.15$ , comparing the method of moments (MoM) and quasi-analytical (QA) solutions. For each method it was verified that as  $\bar{\eta}$  approaches zero, the results approach that of the gap in a perfectly conducting plane in [2]. For  $\bar{\eta}/Z = 0.1$  and  $0.5$  the difference between the peak amplitudes of each method are within 12 percent, and the phase curves show excellent agreement. As expected,  $|P_H|$  is non-zero as  $d$  approaches zero, corresponding to the scattering from a perfectly conducting strip in an impedance plane. Consistent results were verified for  $\bar{\eta}/Z = 0.5$  and  $w/\lambda$  as small as  $0.025$ . As observed in [2], a cyclical behavior exists with increasing gap depth resulting from the periodicity of the impedance looking into the gap.

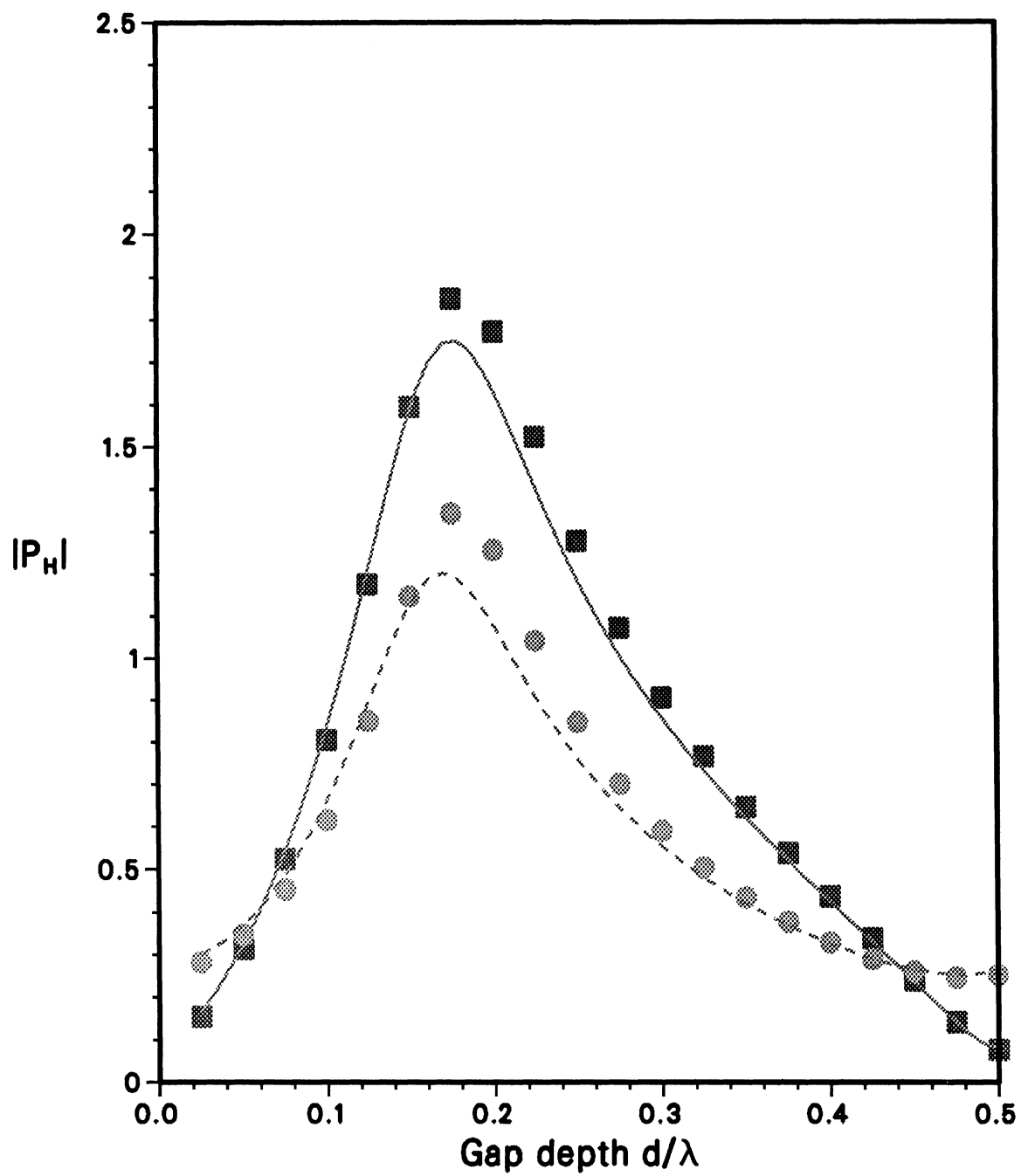


Figure 4. Modulus of the far field amplitude  $P_H$  for a rectangular gap of varying depth  $d_1 = d$  with  $\phi = \phi_0 = \pi/2$  and  $w/\lambda = 0.15$ :

$\eta/Z = 0.1$       ■ MoM, — QA  
 $\eta/Z = 0.5$       ● MoM, - - - QA

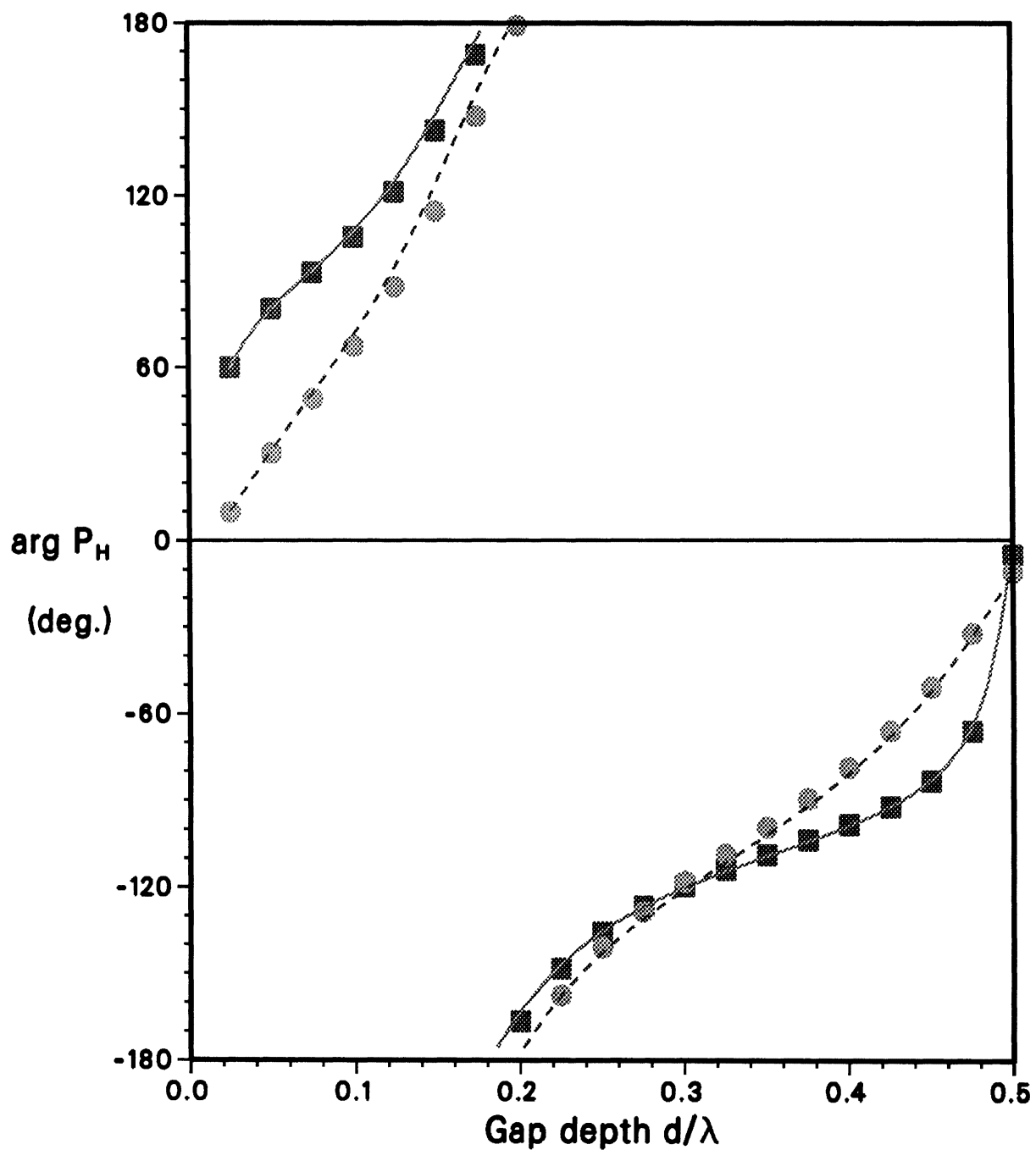


Figure 5. Argument of the far field amplitude  $P_H$  for a rectangular gap of varying depth  $d_1 = d$  with  $\phi = \phi_0 = \pi/2$  and  $w/\lambda = 0.15$ :

$\eta/Z = 0.1$       ■ MoM, — QA  
 $\eta/Z = 0.5$       ● MoM, - - - QA

The far field amplitude backscatter response to the rectangular gap with  $d/\lambda = 0.15$  is contained in Fig. 6, showing excellent agreement between the MoM and QA solutions for all  $\phi$ . Figure 7(a) shows the far field amplitude of the rectangular gap filled with a lossless dielectric having  $\epsilon_r = 2.5$  for  $\bar{\eta}/Z = 0.5$  and 1.0. It was observed that as the relative permittivity of the gap filling was increased from 1, the prediction by the QA method improved, bringing the difference at the peaks within 14 percent for the  $\bar{\eta}/Z = 1.0$  curve shown in the figure. The agreement is improved when some loss is introduced in the dielectric filling, as shown in Fig. 7(b) for  $\epsilon_r = 3 + i0.5$ . For the V-shaped gap of Fig. 2(b), the far field amplitude is presented in Fig. 8 for varying depth. Similar results are expected for gaps of arbitrary geometries, given the appropriate surface impedance  $\eta$  of the gap necessary for the QA solution.

For the E-polarization case, the magnitude and phase are plotted in Figs. 9 and 10 for the rectangular air-filled gap for  $\bar{\eta}/Z = 0.1$  and 0.3. The curves reveal the evanescent nature of the fields in the gap cavity, giving constant values for  $d/\lambda > 0.1$ . In Fig. 9, the greatest deviation occurs as  $d$  approaches zero, for which the MoM predicts a decrease in magnitude. For  $\bar{\eta}/Z = 0.3$ , the difference between the two methods is within 14 percent. The QA method breaks down then for  $\bar{\eta}/Z > 0.4$  since the amplitude continues to increase for all  $d$ . Figure 10 shows that the phase information is lost in using the QA method. A plot of the backscatter from an air-filled rectangular gap with  $d_1/\lambda = 0.2$  is contained in Fig. 11, and Fig. 12 shows the far field amplitude of the scattering from an air-filled V-shaped gap.

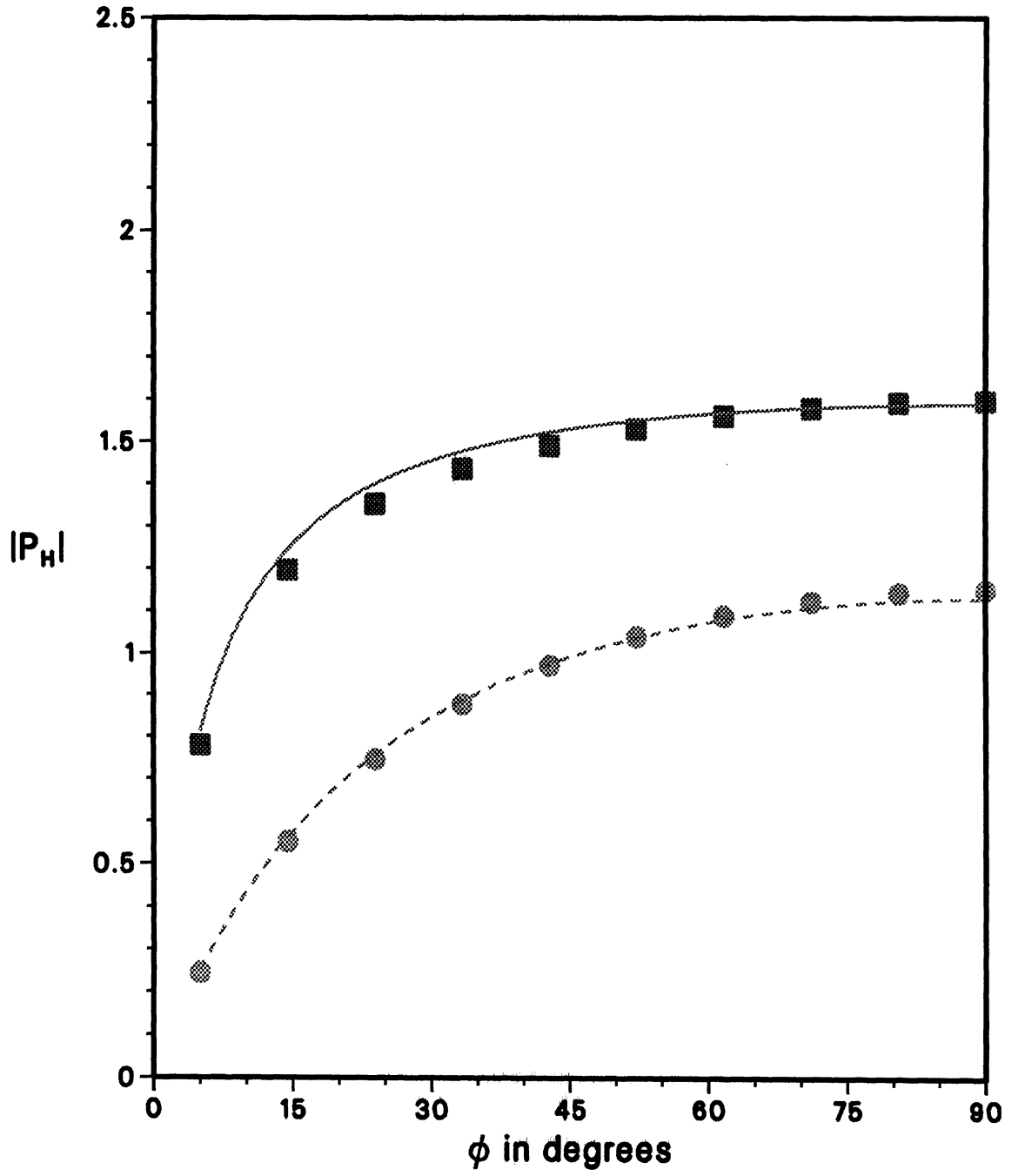


Figure 6. Modulus of the far field amplitude  $P_H$  for the backscatter from a rectangular gap of depth  $d_1/\lambda = 0.15$  with  $\phi_0 = \phi$ ,  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.1$     ■ MoM, — QA  
 $\bar{\eta}/Z = 0.5$     ● MoM, - - - QA



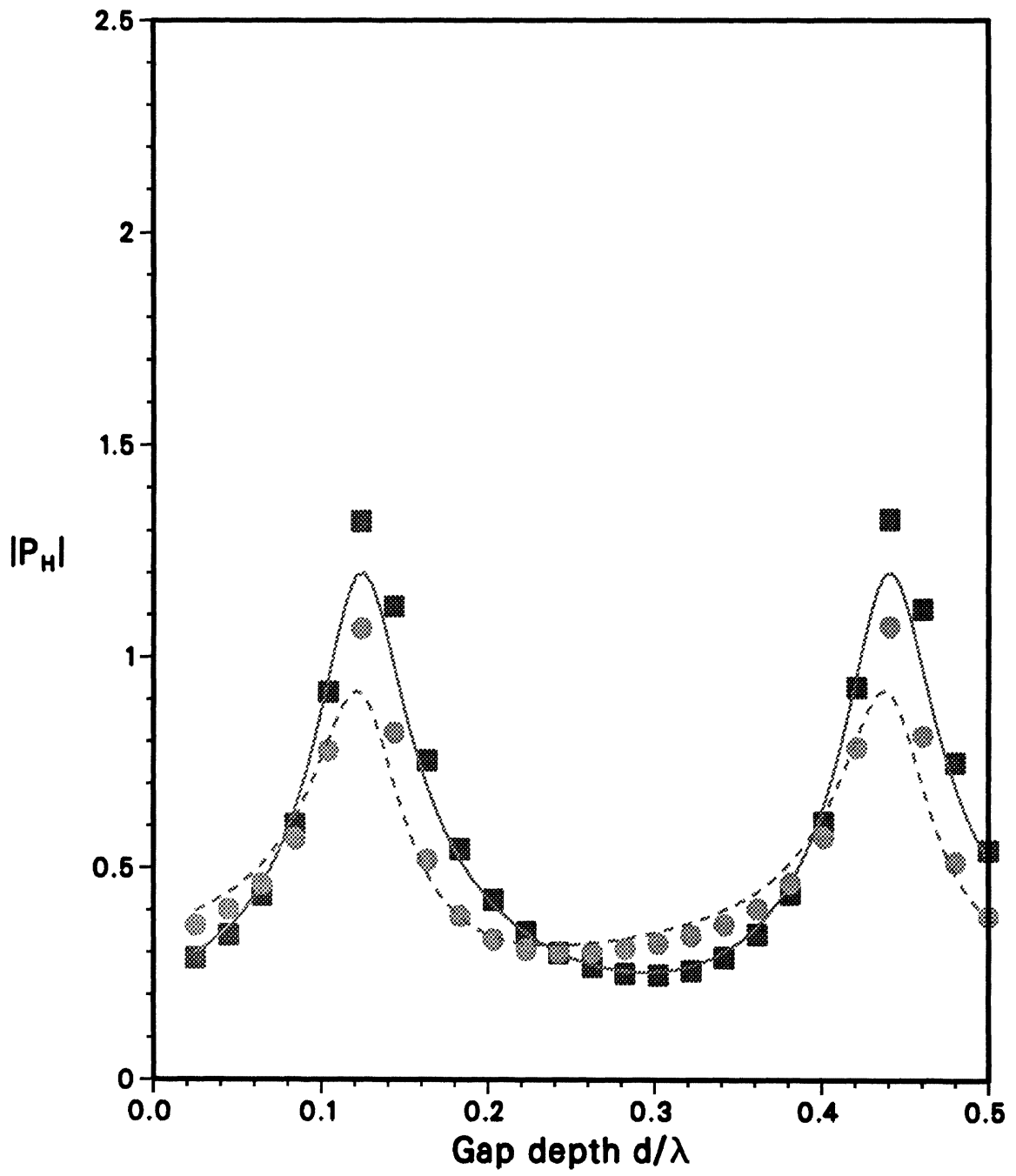


Figure 7(a). Modulus of the far field amplitude  $P_H$  for the material-filled rectangular gap of varying depth  $d_1 = d$  with  $\epsilon_r = 2.5$ ,  $\mu_r = 1$ ,  $\phi = \phi_0 = \pi/2$ , and  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.5$       ■ MoM, — QA  
 $\bar{\eta}/Z = 1.0$       ● MoM, - - - QA

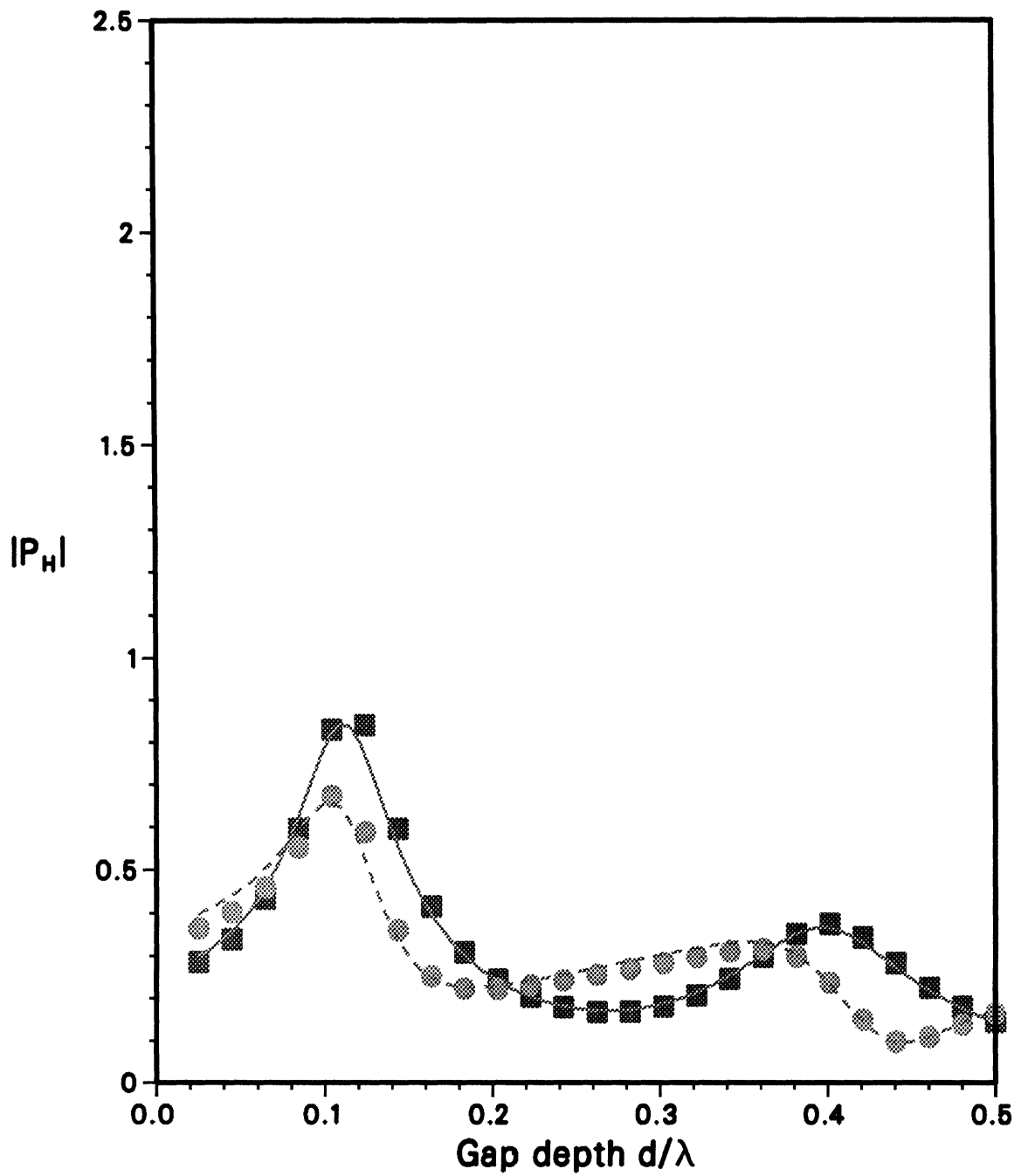


Figure 7(b). Modulus of the far field amplitude  $P_H$  for the material-filled rectangular gap of varying depth  $d_1 = d$  with  $\epsilon_r = 3 + i0.5$ ,  $\mu_r = 1$ ,  $\phi = \phi_0 = \pi/2$ , and  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.5$     ■ MoM, — QA  
 $\bar{\eta}/Z = 1.0$     ● MoM, - - - QA

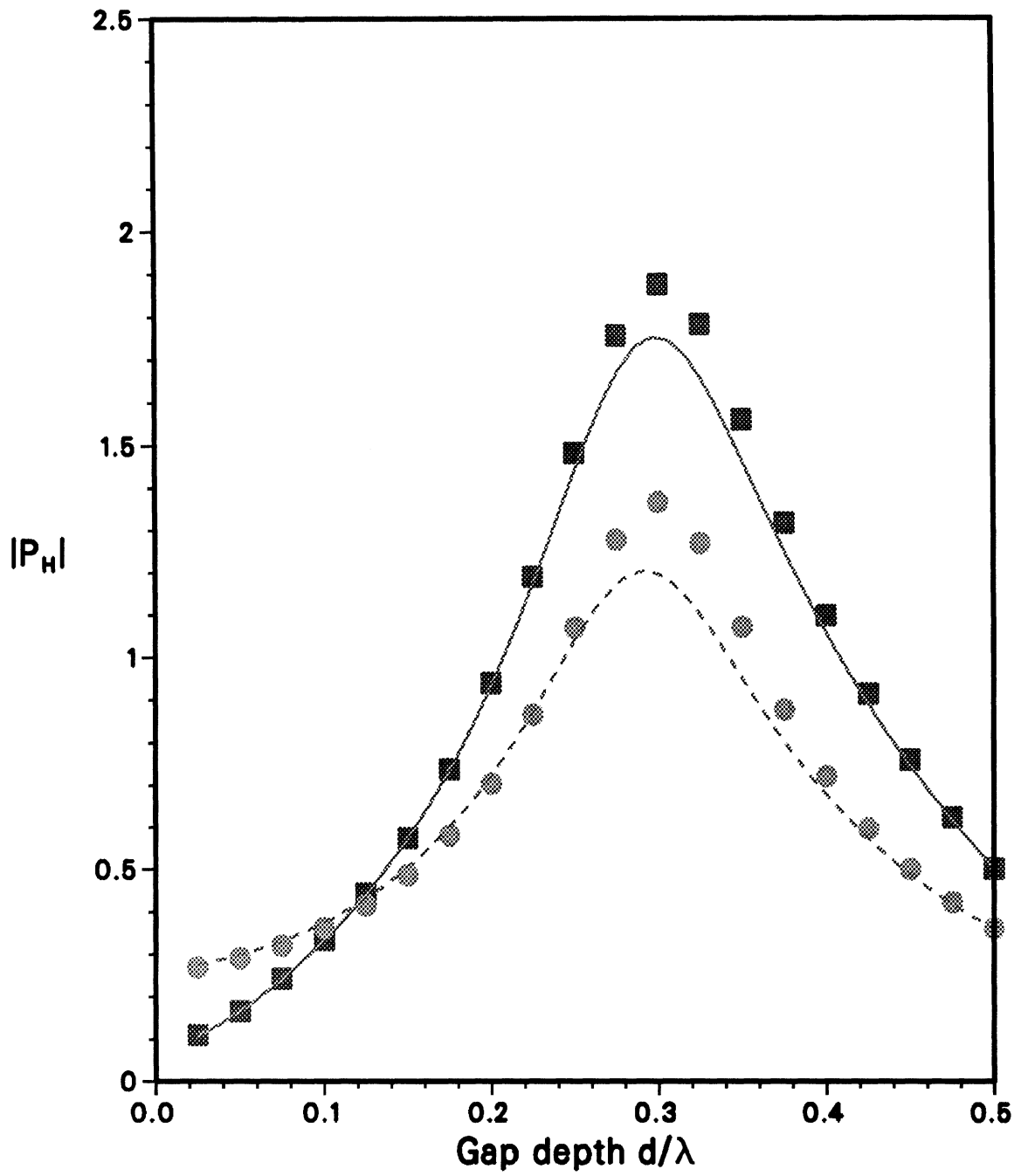


Figure 8. Modulus of the far field amplitude  $P_H$  for an air-filled V-shaped gap of varying depth  $d_1 = d$  with  $\phi = \phi_o = \pi/2$  and  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.1$      $\blacksquare$  MoM, — QA  
 $\bar{\eta}/Z = 0.5$      $\bullet$  MoM, - - - QA

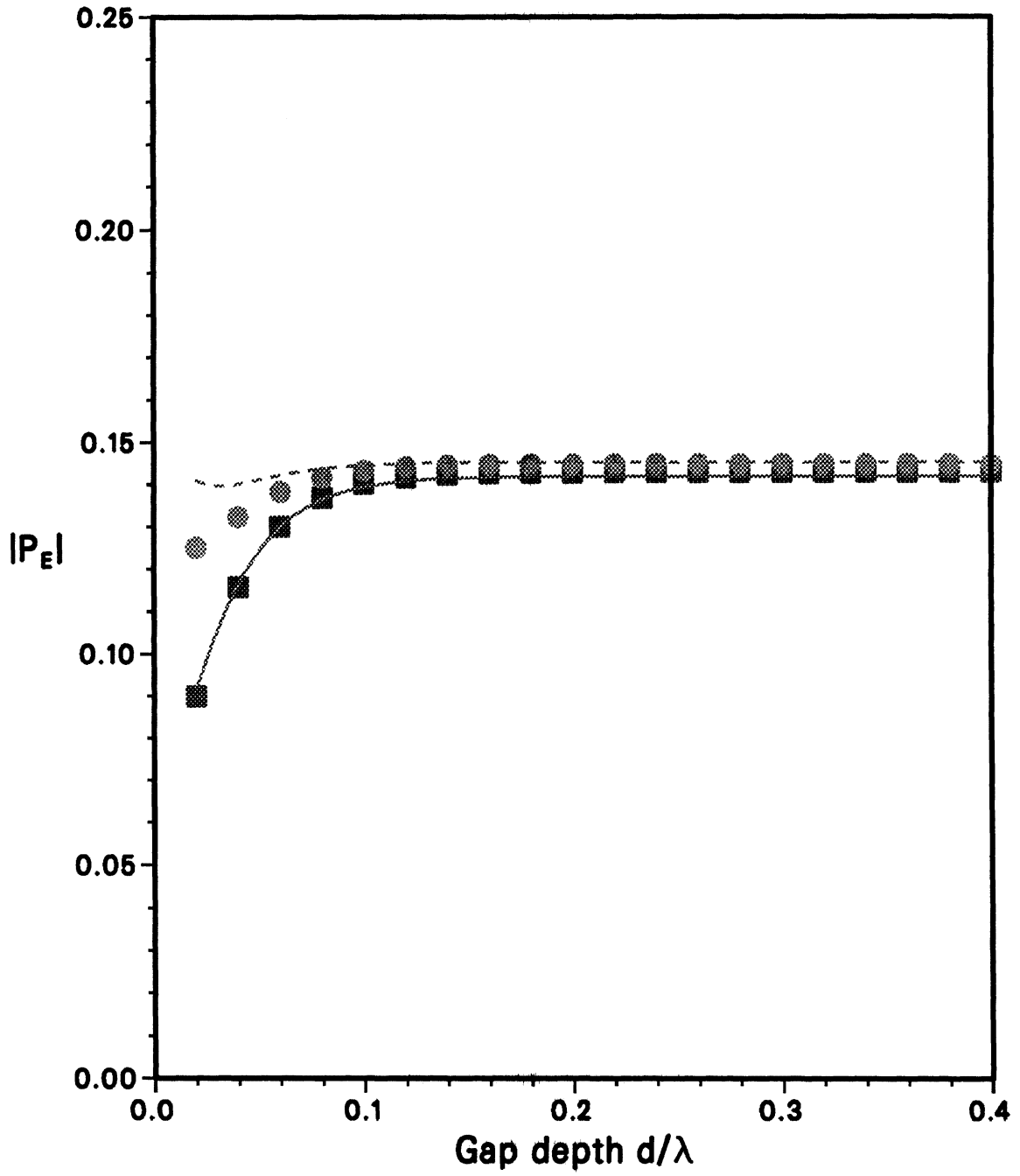


Figure 9. Modulus of the far field amplitude  $P_E$  for an air-filled rectangular gap of varying depth  $d_1 = d$  with  $\phi = \phi_o = \pi/2$  and  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.1$     ■ MoM, — QA  
 $\bar{\eta}/Z = 0.3$     ● MoM, - - - QA

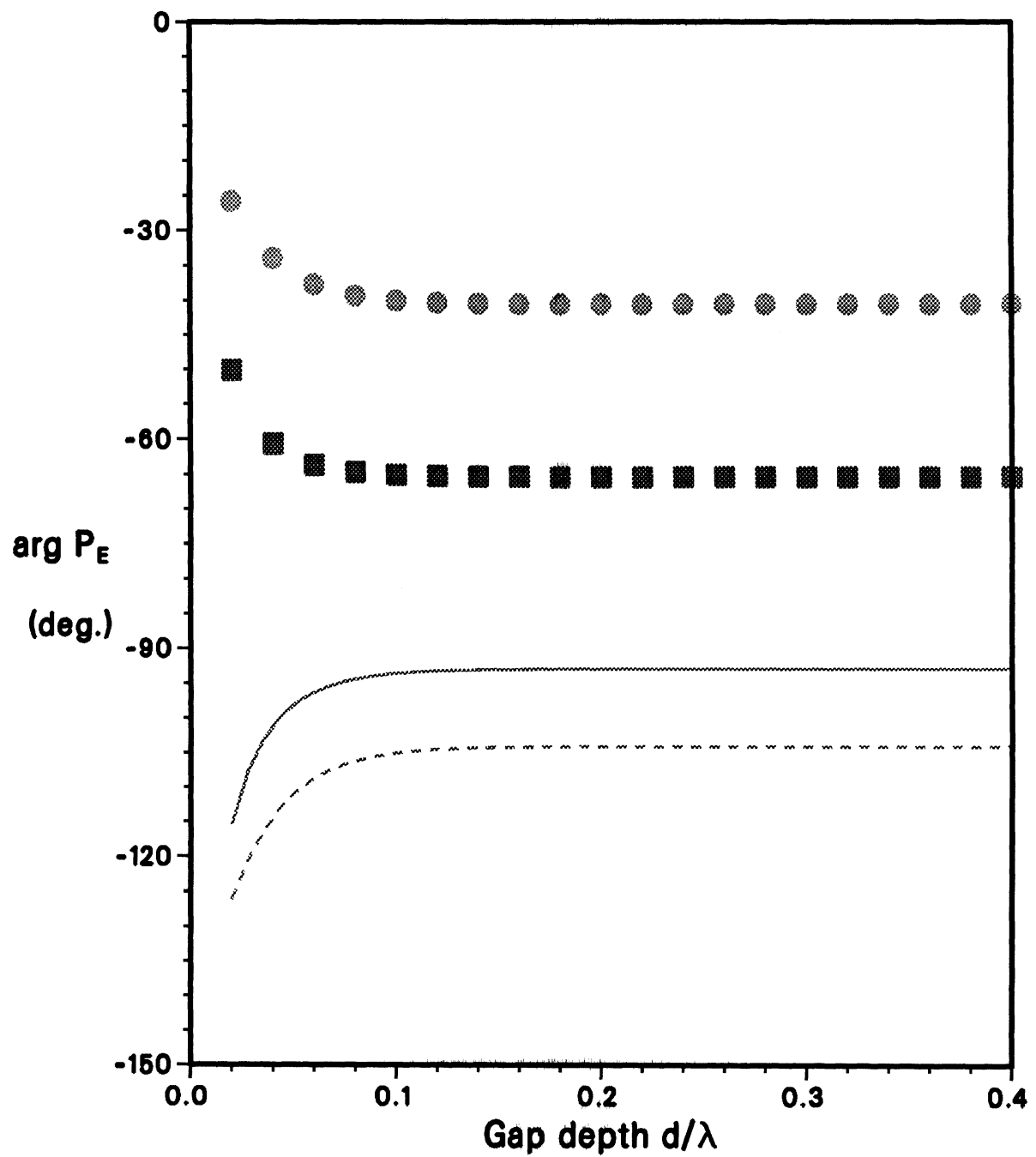


Figure 10. Argument of the far field amplitude  $P_E$  for an air-filled rectangular gap of varying depth  $d_1 = d$  with  $\phi = \phi_o = \pi/2$  and  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.1$       ■ MoM, — QA  
 $\bar{\eta}/Z = 0.3$       ● MoM, - - - QA

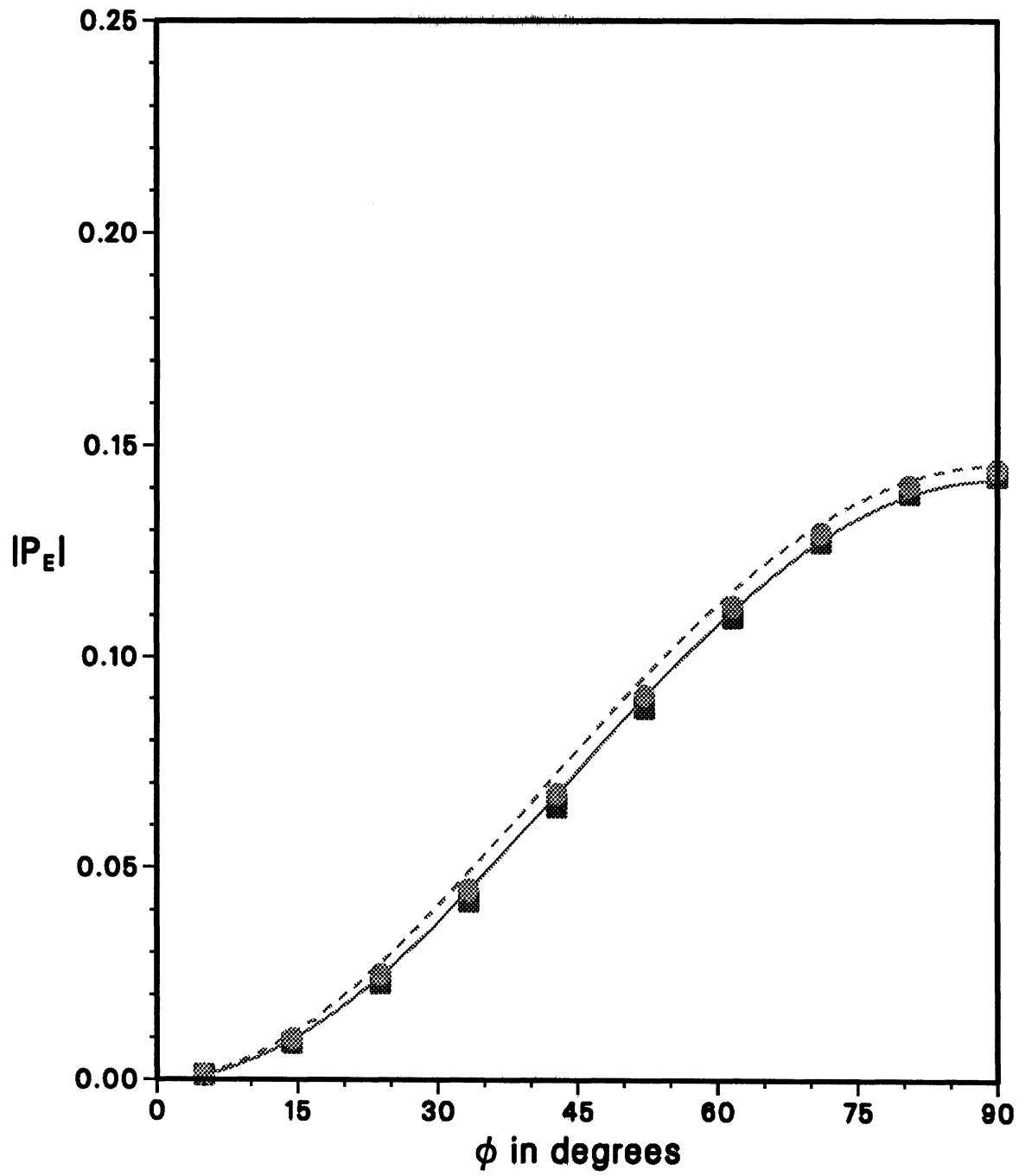


Figure 11. Modulus of the far field amplitude  $P_E$  for the backscatter from a rectangular gap of depth  $d_1/\lambda = 0.2$  with  $\phi_o = \phi$ ,  $w/\lambda = 0.15$ :

$\bar{\eta}/Z = 0.1$     ■ MoM, — QA  
 $\bar{\eta}/Z = 0.3$     ● MoM, - - - QA

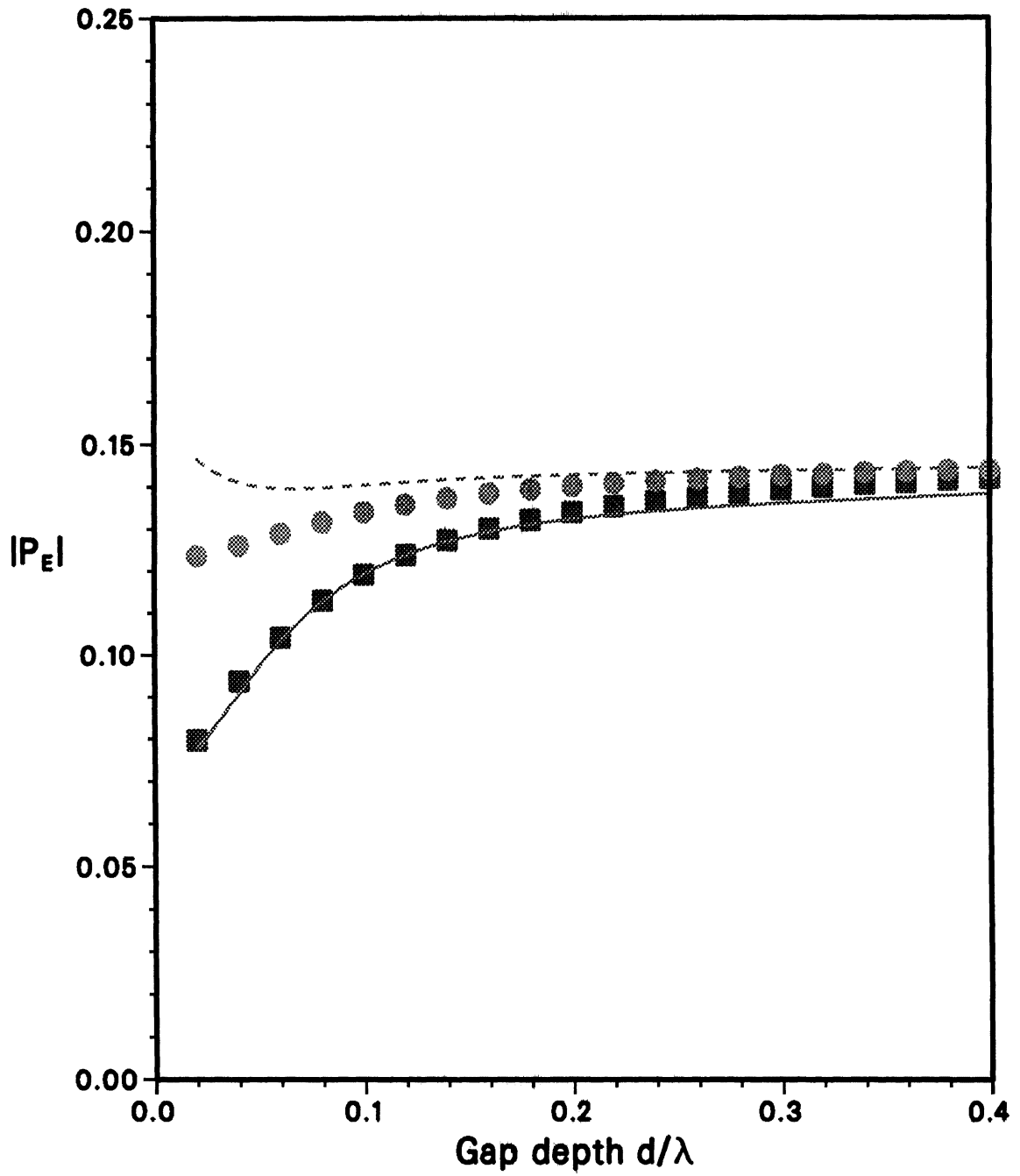


Figure 12. Modulus of the far field amplitude  $P_E$  for an air-filled V-shaped gap of varying depth  $d_1 = d$  with  $\phi = \phi_o = \pi/2$  and  $w/\lambda = 0.15$ :

$\eta/Z = 0.1$       ■ MoM, — QA  
 $\eta/Z = 0.3$       ● MoM, - - - QA

## 5. CONCLUSIONS

The quasi-analytical solution of [2] was based on the low frequency approximation of the integral equations for a constant impedance insert in a perfectly conducting plane applied to a cavity-backed gap in the same such plane. This same solution has been applied to a gap in an impedance plane by modifying the expressions with coefficients dependent on the surface impedance of the plane and the width of the gap. For comparison, a solution was derived by the equivalence principle in conjunction with the half space impedance Green's function, and the exact integral equations were solved by the method of moments. For an air-filled gap, the quasi-analytical solution for the far-field amplitude is within 12 percent for an  $\bar{\eta}/Z \leq 0.5$  for H-polarization and within 14 percent for an  $\bar{\eta}/Z \leq 0.3$  for E-polarization, assuming  $w/\lambda \leq 0.15$ . The results are improved when the gap is material-filled with a relative permittivity greater than 2.5, within 14 percent for  $\bar{\eta}/Z \leq 1$  for H-polarization. The quasi-analytical solution gives excellent agreement with the method of moments results when the gap is filled with a lossy dielectric material. Thus the quasi-analytical solution is a simple and accurate method for determining the scattering from a narrow gap in an impedance plane for surface impedances of small but significant values.



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## APPENDIX A Evaluation of the Half Space Green's Function of an Impedance Plane

The half space Green's function for an impedance plane is given in (3) and, considering the integral, in a more quickly convergent form in (4). For the currents and observation in the gap,  $y' = y = 0$ , and (4) becomes

$$G(x,0;x',0) = \frac{i}{2} \left\{ H_0^{(1)}(k|x-x'|) - \lim_{y \rightarrow 0} \int_0^\infty \tau_H e^{-\tau_H v} H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y+iv)^2} \right) dv \right\}. \quad (A.1)$$

The Hankel function is well defined, but the integral must be analyzed carefully, since its integrand contains a singularity at

$$v = v_0 = |x-x'|$$

in the limit as  $y$  approaches zero. Using the small argument approximation of the Hankel function, the integral about this point is

$$L_H(x,0;x',0|\bar{\eta}) \Big|_{v_0} = \lim_{y \rightarrow 0} \int_{v_0-\Delta v}^{v_0+\Delta v} \tau_H e^{-\tau_H v} \left[ A' + \frac{i2}{\pi} \ell n \sqrt{(x-x')^2 + (y+iv)^2} \right] dv \quad (A.2)$$

where

$$A' = 1 + \frac{i2}{\pi} \left( \gamma + \ell n \frac{k}{2} \right). \quad (A.3)$$

Assuming  $\Delta v$  is small such that the exponential term is nearly constant over  $2\Delta v$ , (A.2) is evaluated analytically in the limit as

$$L_H^o = \tau_H e^{-\tau_H v_0} (L_1 + L_2) \quad (A.4)$$

where

$$L_1 = 2A' \Delta v$$

and

$$L_2 = \frac{i}{\pi} \left\{ (v_o + \Delta v) \left[ \ell n(2v_o \Delta v + \Delta v^2) + i\pi \right] - (v_o - \Delta v) \ell n(2v_o \Delta v - \Delta v^2) \right. \\ \left. - 4\Delta v + v_o \left[ \ell n \left( \frac{2v_o + \Delta v}{2v_o - \Delta v} \right) - i\pi \right] \right\}.$$

Over the other portions of the path of integration,  $L_H$  is evaluated numerically.

For the case  $x = x'$ , we have  $v_o = 0$ , and the evaluation of (A.1) can be done as follows. In the application of the moment method, the path of the integral in (9) is segmented such that the currents are assumed to be constant over each segment  $\Delta x$ . The currents can then be removed from the integral, and with the self-cell, the integration of  $L_H$  over  $\Delta x$  is

$$\int_{x_o - \Delta x/2}^{x_o + \Delta x/2} \lim_{y \rightarrow 0} \left\{ \int_0^{\Delta v} \tau_H e^{-\tau_H v} \left[ A' + \frac{i2}{\pi} \ell n \sqrt{(x_o - x')^2 + (y + iv)^2} \right] dv \right. \\ \left. + \int_{\Delta v}^{\infty} \tau_H e^{-\tau_H v} H_o^{(1)} \left( k \sqrt{(x_o - x')^2 + (y + iv)^2} \right) dv \right\} dx',$$

where  $x_o$  is the midpoint of the self-cell. Evaluating the first integral within the brackets analytically, the self-cell expression of  $L_H$  becomes

$$S_H(\Delta x | \bar{\eta}) = L_H^{ox} + 2 \int_{x_o}^{x_o + \Delta x/2} \lim_{y \rightarrow 0} \int_{\Delta v}^{\infty} \tau_H e^{-\tau_H v} H_o^{(1)} \left( k \sqrt{(x_o - x')^2 + (y + iv)^2} \right) dv dx' \quad (A.5)$$

where

$$L_H^{ox} = \tau_H e^{-\tau_H \frac{\Delta v}{2}} (L_3 + L_4) \quad (A.6)$$

with

$$L_3 = \left( A' - \frac{i2}{\pi} \right) \Delta x \Delta v$$

and

$$L_4 = \frac{i}{\pi} \Delta x \left\{ \Delta v \ln \left( \frac{\Delta x^2}{4} - \Delta v^2 \right) - 2\Delta v + \frac{\Delta x}{2} \ln \left( \frac{\Delta x + 2\Delta v}{\Delta x - 2\Delta v} \right) \right. \\ \left. - \left( \frac{\Delta x}{4} - \frac{\Delta v^2}{\Delta x} \right) \left[ \ln \left( \frac{\Delta x + 2\Delta v}{\Delta x - 2\Delta v} \right) + i\pi \right] + \frac{\Delta v}{2} + i\pi \frac{\Delta x}{4} \right\}.$$

The double integral in (A.5) is evaluated numerically. The self-cell expression of the Hankel function in (A.1) is well known and is given in Appendix B as needed.

For the E-polarization case, the derivatives of the Green's function must be considered. From (18), and using the form in (4) for (13),

$$\lim_{\substack{y' \rightarrow 0 \\ y \rightarrow 0}} \frac{\partial^2}{\partial y \partial y'} G(x, y; x', y') = \frac{i}{2} \lim_{y \rightarrow 0} \left( k^2 + \frac{\partial^2}{\partial x^2} \right) \int_0^\infty \tau_E e^{-\tau_E y} H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y+iv)^2} \right) dv \quad (A.7)$$

The order of the differential operator is reduced by applying the integral shown in (18) over the segment  $\Delta x$ . Given the endpoints  $x_1, x_2$  of the segment, the integral  $L_E$  of (A.7) becomes

$$\int_{x_1}^{x_2} \int_0^\infty \tau_E e^{-\tau_E y} H_0^{(1)} \left( k \sqrt{(x-x')^2 + (y+iv)^2} \right) dv dx' \\ + \frac{1}{k} \int_0^\infty \tau_E e^{-\tau_E y} \left[ \frac{(x-x_2)}{\sqrt{(x-x_2)^2 + (y+iv)^2}} H_1^{(1)} \left( k \sqrt{(x-x_2)^2 + (y+iv)^2} \right) \right. \\ \left. - \frac{(x-x_1)}{\sqrt{(x-x_1)^2 + (y+iv)^2}} H_1^{(1)} \left( k \sqrt{(x-x_1)^2 + (y+iv)^2} \right) \right] dv, \quad (A.8)$$

in the limit as  $y$  approaches zero, where  $k^2$  has been factored out. For  $v_0$  not equal to zero, the singularity in the first integral expression of (A.8) is handled in the same way as that for the H-polarization case, giving

$$L_E^0 = \tau_E e^{-\tau_E v_0} (L_1 + L_2) \quad (A.9)$$

with  $L_1$  and  $L_2$  given above. The integration is done numerically over the remaining path, as well as with respect to  $x'$ . Let  $L_E'(x, \Delta x' | \bar{\eta})$  denote the second integral expression in (A.8). The contribution from the singularity is, using the small argument approximation of the first order Hankel function,

$$L_E^1 = \frac{\tau_E}{k} \left\{ e^{-\tau_E v_{o2}} \left[ k(x-x_2) \Delta v + \frac{i}{\pi k} \left( \ell n \frac{2v_{o2} - \Delta v}{2v_{o2} + \Delta v} + i\pi \right) \frac{(x-x_2)}{|x-x_2|} \right] - e^{-\tau_E v_{o1}} \left[ k(x-x_1) \Delta v + \frac{i}{\pi k} \left( \ell n \frac{2v_{o1} - \Delta v}{2v_{o1} + \Delta v} + i\pi \right) \frac{(x-x_1)}{|x-x_1|} \right] \right\} \quad (A.10)$$

where

$$v_{o1} = |x-x_1|, \quad v_{o2} = |x-x_2|.$$

Considering the integrations of (A.8) over the self-cell,

$$S_E(\Delta x | \bar{\eta}) = L_E^{ox} + 2 \int_{x_0}^{x_0 + \Delta x/2} \lim_{y \rightarrow 0} \int_{\Delta v}^{\infty} \tau_E e^{-\tau_E v} H_0^{(1)} \left( k \sqrt{(x_0 - x')^2 + (y + iv)^2} \right) dv dx' + L_E^{1x} - \frac{1}{k} \lim_{y \rightarrow 0} \int_{\Delta v}^{\infty} \tau_E e^{-\tau_E v} \frac{\Delta x}{\sqrt{(\Delta x/2)^2 + (y + iv)^2}} H_1^{(1)} \left( k \sqrt{(\Delta x/2)^2 + (y + iv)^2} \right) dv \quad (A.11)$$

where the analytical expressions are

$$L_E^{ox} = \tau_E e^{-\tau_E \frac{\Delta v}{2}} (L_3 + L_4) \quad (A.12)$$

and

$$L_E^{1x} = \tau_E e^{-\tau_E \frac{\Delta v}{2}} \frac{2}{k} \left[ -k \frac{\Delta x}{4} \Delta v + \frac{i}{\pi k} \ell n \left( \frac{\Delta x + 2\Delta v}{\Delta x - 2\Delta v} \right) \right]. \quad (A.13)$$

## APPENDIX B Moment Method Solution of the Coupled Integral Equations

The integral equation pairs given by (7),(9) and (18),(19) are solved by the moment method [2,Appendix A]. Using pulse basis functions in the moment method, the aperture A and the cavity walls S shown in Fig. 1 are segmented into N cells of size  $\Delta s$ . The magnetic and electric currents are assumed to be constant over each of these segments. When the integrations of the coupled equations are taken over each segment, the current expressions can be removed as constants from the integrals. With the contour of integration discretized, the  $(x',y')$  coordinates become  $(x_i,y_i)$ ,  $i = 1, \dots, N$ , which describe the location of each of the segments. The Green's functions can then be expressed in terms of rotated coordinates  $(s,n)$  for the observation position and  $(s_i,n_i)$  for each segment or source position since the integration is with respect to the tangential vector  $\hat{s}$  as shown in Fig. 1.

The expressions for the numerical solution of the coupled equations are developed in the following sections for the H- and E-polarization cases. Applying point matching, the magnetic and electric currents in the aperture and on the cavity walls are determined, and the far field amplitude is calculated.

### A.1 H-Polarization

For the discretized contour of integration, (7) and (9) become

$$J_s(s,n) = \frac{kY}{2} \epsilon_r \sum_{i=1}^M J_z^*(s_i) \int_{\Delta s_i} H_o^{(1)} \left( k_1 \sqrt{(s-s_i)^2 + n^2} \right) ds_i \\ + \frac{ik_1}{2} \sum_{i=1}^N J_s(s_i,n_i) \int_{\Delta s_i} \sin \beta_i H_1^{(1)} \left( k_1 \sqrt{(s-s_i)^2 + (n-n_i)^2} \right) ds_i \quad (B.1)$$

$$J_s(s) = \frac{2 \sin \phi_0}{\bar{\eta} Y + \sin \phi_0} e^{-iks \cos \phi_0} - \frac{kY}{2} \sum_{i=1}^M \left[ J_z^*(s_i) - \bar{\eta} J_s(s_i) \right] \int_{\Delta s_i} \left[ H_o^{(1)}(k |s-s_i|) - L_H(s,0;s_i,0) \right] ds_i \quad (B.2)$$

where M are the number of segments across the aperture and

$$\sin \beta_i = \frac{(n-n_i)}{\sqrt{(s-s_i)^2 + (n-n_i)^2}}. \quad (B.3)$$

Applying point matching over the N segments of the aperture and cavity walls,

$$\sum_{i=1}^N I_i \left[ -1 \Big|_{j=i} + \frac{ik_1}{2} \int_{\Delta s_i} \sin \beta_{j,i} H_1^{(1)}(k_1 R_{j,i}) ds_i \right] + \frac{kY}{2} \epsilon_r \sum_{i=N+1}^{N+M} I_i \int_{\Delta s_i} H_o^{(1)}(k_1 R_{j,i}) ds_i = 0 \quad (B.4)$$

$$\sum_{i=1}^M I_i \left[ 1 \Big|_{j=i} - \frac{kY}{2} \bar{\eta} \int_{\Delta s_i} \left[ H_o^{(1)}(k R_{j,i}) - L_H(s_j,0;s_i,0) \right] ds_i \right] + \frac{kY}{2} \sum_{i=N+1}^{N+M} I_i \int_{\Delta s_i} \left[ H_o^{(1)}(k R_{j,i}) - L_H(s_j,0;s_i,0) \right] ds_i = \frac{2 \sin \phi_0}{\bar{\eta} Y + \sin \phi_0} e^{iks_j \cos \phi_0} \quad (B.5)$$

where

$$R_{j,i} = \sqrt{(s_j-s_i)^2 + (n_j-n_i)^2}. \quad (B.6)$$

The coordinate  $(s_j, n_j)$  is the observation position at the midpoint of the  $j^{\text{th}}$  segment. Hence, for  $i, j = 1, \dots, M, N+1, \dots, N+M$ , the segments are located in the aperture, and for  $i, j = M+1, \dots, N$ , the segments are located on the cavity walls.  $I_i$  in (B.4) and (B.5) are the electric currents, for  $i = 1, \dots, N$ , and the magnetic currents,  $i = N+1, \dots, N+M$ , to be determined.

In matrix form, (B.4) and (B.5) become

$$[Z_{j,i}] [I_i] = [V_j] . \quad (B.7)$$

The impedance matrix is given as

$$[Z_{j,i}] = \begin{bmatrix} Z_{e1} & \vdots & Z_{m1} \\ \dots\dots\dots & & \dots\dots\dots \\ Z_{e2} & \vdots & Z_{m2} \end{bmatrix} \quad (B.8)$$

where the sets of elements are as follows:

$$Z_{e1} = \begin{cases} \frac{ik_1}{2} \int_{\Delta s_i} \sin \beta_{j,i} H_1^{(1)}(k_1 R_{j,i}) ds_i & j \neq i \\ -1 & j = i \end{cases} \quad (B.9)$$

for  $i = 1, \dots, N$  and  $j = 1, \dots, N$ ;

$$Z_{m1} = \begin{cases} \frac{kY}{2} \epsilon_r \int_{\Delta s_i} H_o^{(1)}(k_1 R_{j,i}) ds_i & j \neq i-N \\ \frac{kY}{2} \epsilon_r 2(s_i - s_j) \left[ \frac{i2}{\pi} \ell n(R_{j,i}) - 1 + A'_1 \right] & j = i-N \end{cases} \quad (B.10)$$

for  $i = N+1, \dots, N+M$  and  $j = 1, \dots, N$ ;

$$Z_{e2} = \begin{cases} -\frac{kY}{2} \bar{\eta} \int_{\Delta s_i} \left[ H_o^{(1)}(k R_{j,i}) - L_H(s_j, 0; s_i, 0) \right] ds_i & j \neq i+N \\ 1 - \frac{kY}{2} \bar{\eta} \left[ 2(s_i - s_j) \left( \frac{i2}{\pi} \ell n(R_{j,i}) - 1 + A' \right) - S_H(\Delta s_i | \bar{\eta}) \right] & j = i+N \end{cases} \quad (B.11)$$

for  $i = 1, \dots, N$  and  $j = N+1, \dots, N+M$ ;



$$Z_{m2} = \begin{cases} \frac{kY}{2} \int_{\Delta s_i} \left[ H_o^{(1)}(k R_{j,i}) - L_H(s_j, 0; s_i, 0) \right] ds_i & j \neq i \\ \frac{kY}{2} \left[ 2(s_i - s_j) \left( \frac{i2}{\pi} \ln(R_{j,i}) - 1 + A' \right) - S_H(\Delta s_i | \bar{\eta}) \right] & j = i \end{cases} \quad (B.12)$$

for  $i = N+1, \dots, N+M$  and  $j = N+1, \dots, N+M$ . In (B.10), the expression for  $A'_1$  is

$$A'_1 = 1 + \frac{i2}{\pi} \left( \gamma + \ln \frac{k_1}{2} \right),$$

and in (B.11) and (B.12),  $S_H$  is given by (A.5). For the self-cells in (B.10) to (B.12),  $s_i$  is taken to be the endpoint of the  $i^{\text{th}}$  segment. The self-cell expressions were derived analytically, and a numerical integration is applied to the other segments. In the case of the V-shaped gap, the adjacent cells needed to be evaluated in the vicinity of  $y = -d$ , for  $R_{j,i}$  less than one cell size, and these expressions can be found in Appendix A of [2].

The source matrix is given by

$$V_j = \begin{cases} 0 & j = 1, \dots, N \\ \frac{2 \sin \phi_o}{\bar{\eta} Y + \sin \phi_o} e^{-iks_j \cos \phi_o} & j = N+1, \dots, N+M \end{cases}, \quad (B.13)$$

and the currents  $I_i$  are determined by solving (B.7), given that  $[Z_{j,i}]$  is nonsingular. From (11), the far field amplitude at the angle  $\phi$  is now

$$P_H(\phi, \phi_o) = -\frac{kY}{2} \frac{1}{\bar{\eta} Y + 1} \sum_{i=1}^M \left( I_{i+N} - \bar{\eta} I_i \right) \int_{\Delta x_i} e^{-ikx_i \cos \phi} dx_i. \quad (B.14)$$

## B.2 E-Polarization

The integral equation pair given by (18) and (19) was solved in the same manner as described for the H-polarization case. The elements of the impedance matrix defined in (B.8) for the E-polarization case are as follows:

$$Z_{e1} = \begin{cases} \frac{kZ}{2} \mu_r \int_{\Delta s_i} H_o^{(1)}(k_1 R_{j,i}) ds_i & j \neq i \\ \frac{kZ}{2} \mu_r 2(s_i - s_j) \left[ \frac{i2}{\pi} \ln(R_{j,i}) - 1 + A'_1 \right] & j = i \end{cases} \quad (B.15)$$

for  $i = 1, \dots, N$  and  $j = 1, \dots, N$ ;

$$Z_{m1} = \begin{cases} -\frac{ik_1}{2} \int_{\Delta s_i} \frac{y_j}{R_{j,i}} H_1^{(1)}(k_1 R_{j,i}) ds_i & j \neq i-N \\ -1 & j = i-N \end{cases} \quad (B.16)$$

for  $i = N+1, \dots, N+M$  and  $j = 1, \dots, N$ ;

$$Z_{e2} = \begin{cases} -\frac{kY}{2} \bar{\eta} \left[ \int_{\Delta s_i} L_E(s_j, 0; s_i, 0 | \bar{\eta}) ds_i + L'_E(s_j, \Delta s_i | \bar{\eta}) \right] & j \neq i+N \\ -1 - \frac{kY}{2} \bar{\eta} S_E(\Delta s_i | \bar{\eta}) & j = i+N \end{cases} \quad (B.17)$$

for  $i = 1, \dots, N$  and  $j = N+1, \dots, N+M$ ;

$$Z_{m2} = \begin{cases} -\frac{kY}{2} \left[ \int_{\Delta s_i} L_E(s_j, 0; s_i, 0 | \bar{\eta}) ds_i + L'_E(s_j, \Delta s_i | \bar{\eta}) \right] & j \neq i \\ -\frac{kY}{2} S_E(\Delta s_i | \bar{\eta}) & j = i \end{cases} \quad (B.18)$$

for  $i = N+1, \dots, N+M$  and  $j = N+1, \dots, N+M$ . In the self-cell expression of (B.15),  $s_i$  is

evaluated at the endpoint of the  $i^{\text{th}}$  segment. In (B.17) and (B.18),  $S_E$  is given by (A.11).

The source matrix is

$$V_j = \begin{cases} 0 & j = 1, \dots, N \\ \frac{2Y}{\bar{\eta}Y + \csc\phi_0} e^{-iks_j \cos\phi_0} & j = N+1, \dots, N+M \end{cases} \quad (\text{B.19})$$

Given that  $[Z_{j,i}]$  is nonsingular, the currents  $I_i$  can be determined, and the far field amplitude is calculated from the expression

$$P_E(\phi, \phi_0) = -\frac{k}{2} \frac{\sin\phi}{\bar{\eta}Y + 1} \sum_{i=1}^M (I_{i+N} + \bar{\eta} I_i) \int_{\Delta x_i} e^{-ikx_i \cos\phi} dx_i \quad (\text{B.20})$$

### B.3 Program Listing

The expressions for the impedance and source matrices and the far field amplitude were programmed for solution, as contained in the program listing of IMP.FTN below. The subroutines called by the program in addition to those listed are contained in the file GAPSUB.FTN of Appendix A of [2].

The numerical integration is done for the appropriate segments using Simpson's three-point composite integration over each segment. A segment size of  $\Delta s/\lambda = 0.01$  was used for the results of Figs. 4 to 12. The numerical integration implemented for  $L_H$  and  $L_E$  is the Gauss-quadrature technique, and convergence was verified for  $\bar{\eta}/Z \leq 2$ . As listed, the program calculates the far field amplitude as a function of the gap depth using (B.14) for H-polarization and (B.20) for E-polarization.



```

c...Prompting user for input data
call gaprom(iprg)

if(EorH .eq. 1) Epol=.true.
phio=phio*pi/180.0
phi=phi*pi/180.0
drat=dStp(1)/d
k1=k*csqrt(er*ur)
A=2*(clog(k1/2)+gam-ci*pi/2)
if(almag(er) .ne. 0.0) Lossy=.true.

dmin=.025
if(Epol) dmin=0.02
dmax=d
if(noIter .ne. 1)then
  dstep=(dmax-dmin)/(noIter-1)
  d=dmin
endif
DO 700 iter=1,noIter

c...Determining coordinates of corner points given gap type
if(igap .eq. 1)then
c  RECTANGULAR
  noS=3
  q(3,1)=w/2
  q(3,2)=-d
  q(4,1)=-w/2
  q(4,2)=-d
else if(igap .eq. 2)then
c  L-SHAPED
  noS=7
  dStp(1)=d*drat
  dStp(2)=d*(1-drat)
  q(3,1)=w/2
  q(3,2)=-dStp(1)
  q(8,1)=-w/2
  q(8,2)=-dStp(1)
  q(4,1)=w/2+wStp(2)
  q(4,2)=-dStp(1)
  q(7,1)=-w/2-wStp(3)
  q(7,2)=q(4,2)
  q(5,1)=q(4,1)
  q(5,2)=q(4,2)-dStp(2)
  q(6,1)=q(7,1)
  q(6,2)=q(5,2)
else if(igap .eq. 3)then
c  V-SHAPED
  noS=2
  q(3,1)=0.0
  q(3,2)=-d
  adj=0.75*maxC
else if(igap .eq. 4)then
c  T-SHAPED
  noS=5
  dStp(1)=d*drat
  dStp(2)=d*(1-drat)
  q(3,1)=w/2
  q(3,2)=-dStp(1)
  q(4,1)=w/2+wStp(2)
  q(4,2)=-dStp(1)
  q(5,1)=q(4,1)
  q(5,2)=q(4,2)-dStp(2)
  q(6,1)=-w/2
  q(6,2)=q(5,2)
endif
c...Corner points of gap at y=0
q(1,1)=-w/2
q(1,2)=0.0
q(2,1)=w/2
q(2,2)=0.0
q(noS+2,1)=-w/2
q(noS+2,2)=0.0

c*****
c***** Current segment locations (xi,yi) *****
c*****
N=0
do 175 l=1,noS+1
c...Size (length) of lth side of gap
szSd(l)=sqrt((q(l+1,1)-q(l,1))**2
&          +(q(l+1,2)-q(l,2))**2)
c..Angle of rotation for each side with respect to x axis
if(q(l+1,1) .lt. q(l,1))then
  psi(l)=asin((q(l,2)-q(l+1,2))/szSd(l))
  neg(l)=.true.
else
  psi(l)=asin((q(l+1,2)-q(l,2))/szSd(l))
  neg(l)=.false.
endif

szN(l)=int(szSd(l)/maxC)+1
N=N+szN(l)
spaceX=(q(l+1,1)-q(l,1))/szN(l)
spaceY=(q(l+1,2)-q(l,2))/szN(l)
posiX=q(l,1)
posiY=q(l,2)

```

```

c...ENDPOINTS of each segment are p, MIDPOINTS are m in (x,y)
c   coordinates
      do 170 i=N-szN(1)+1,N
        p(i,1)=posiX
        p(i,2)=posiY
        m(i,1)=posiX+spaceX/2.0
        m(i,2)=posiY+spaceY/2.0
        posiX=posiX+spaceX
        posiY=posiY+spaceY
170   continue
175   continue
      p(N+1,1)=-w/2
      p(N+1,2)=0.0
c...Number of segments in the aperture
      gN=szN(1)
c...Number of current coefficients to be calculated
      NgN=N+gN
      print *, ' d = ',d,' N = ',N,' gN = ',gN
c...Initializing matrices to zero
      do 190 j=1,NgN
        do 180 i=1,NgN
          Z(j,i)=czero
180   continue
          Vj(j)=czero
190   continue

c*****
c*****      Impedance, Source,      *****
c*****      and Current Matrices      *****
c*****

C
C   VARIABLES:
C   H0o,H0   Hankel function of zero order in free
C             space and in material er, respectively.
C   H1o,H1   Hankel function of first order in free
C             space and in material er, respectively.
C   iH0o,    Evaluation of the integral expression of
C   iH1o     the half space Green's function.
C   Green's Function integrals:
C   LH0      Integral of H0.
C   LH1      Integral of H1 for H-pol, of dH1/dy for
C             E-pol.
C   LH0o     Integral of H0o and iH0o for H-pol and
C             iH0o for E-pol.
C   LH1o     Integral of iH1o for E-pol.
C   aLH0     Analytical integral of H0o for evaluation
C             of adjacent cells for LH2, E-pol case.
C
C   The integration is done one side at a time, for j=1,...,N,
C   in the clockwise direction, starting at (x,y)=(-w/2,0).
C
      istart=1
      istop=szN(1)
c...Source point is i of the lth side, observation point is j
      do 230 l=1,nos+1
        do 220 i=istart,istop
          do 210 j=1,N
c      Coordinate rotation for observation point
          sj=m(j,1)*cos(psi(1))+m(j,2)*sin(psi(1))
          nj=m(j,1)*sin(psi(1))-m(j,2)*cos(psi(1))
          if(neg(l))then
            sj=-sj
            nj=-nj
          endif
c...Integration over ith segment
          LH0=czero
          LH1=czero
          LH0o=czero
          LH1o=czero
c      Magnitude between midpoints Rm=|r-r'|
          Rm=sqrt((m(j,1)-m(i,1))**2+(m(j,2)-m(i,2))**2)
          if(j.eq.1.or.Rm.le.adj)then
c      SMALL ARGUMENT APPROXIMATION integral for self-cell
c      and adjacent cells
            do 200 ip=i+1,i,-1
c      Coordinate rotation for source segment points
              si=p(ip,1)*cos(psi(1))+p(ip,2)*sin(psi(1))
              ni=p(ip,1)*sin(psi(1))-p(ip,2)*cos(psi(1))
              if(neg(l))then
                si=-si
                ni=-ni
              endif
              R=sqrt((sj-si)**2+(nj-ni)**2)
              if(j.eq.1.or.abs(nj-ni).eq.0.0)then
                tanf=pi/2
                absf=1.0
              else
                tanf=atan((si-sj)/abs(nj-ni))
                absf=abs(nj-ni)
              endif
              LH1=-k1**2/2*(nj-ni)*si
              & +ci*2./pi*(nj-ni)/absf*tanf-LH1
              & LH0=ci/pi*(2*(si-sj)*log(R)
              & -(2-A)*si+2.0*abs(nj-ni)*tanf)-LH0

```

```

        LH0o=ci/pi*(2*(si-sj)*log(R)
&         -(2-Ao)*si+2.0*abs(nj-ni)*tanf)-LH0o
        if(j .eq. i) GOTO 202
200      continue
202      if(j .eq. i) then
        LH0=2*(LH0+ci/pi*(2-A)*sj)
        if(i .le. gN .and. j .le. gN) then
          if(i .eq. 1 .and. iter .eq. 1) then
            iH1o=abs(sj-si)
            call simpGF(0., etab, Epol, iH0o, iH1o)
            cself0=iH0o
            cself1=iH1o
          endif
          if(Epol) then
            LH1o=cself0+cself1
          else
            LH0o=2*(LH0o+ci/pi*(2-Ao)*sj)-cself0
          endif
        endif
      endif
    else
c      SIMPSON'S THREE POINT COMPOSITE INTEGRATION
      do 204 ip=i+1,i,-1
c      Coordinate rotation for source segment endpoints
      si=p(ip,1)*cos(psi(1))+p(ip,2)*sin(psi(1))
      ni=p(ip,1)*sin(psi(1))-p(ip,2)*cos(psi(1))
      if(neg(l)) then
        si=-si
        ni=-ni
      endif
      stepS=si
c      HANKEL FUNCTION evaluation at endpoints of segment
      R=sqrt((sj-si)**2+(nj-ni)**2)
      if(Lossy) then
        ckrho=k1*R
        call cHank(ckrho,2,H0,H1)
      else
        krho=Real(k1)*R
        call Hankz1(krho,2,H0,H1)
      endif
      if(i .le. gN .and. j .le. gN) then
        krho=k*R
        call Hankz1(krho,2,H0o,H1o)
        if(i .eq. 1 .and. iter .eq. 1) then
          call impGF(krho, etab, Epol, ctemp0, ctemp1)
          if(i .eq. ip) then
            GF0a(j)=ctemp0
            iH0o=GF0a(j)
            GF1a(j)=ctemp1
            iH1o=GF1a(j)
          else
            GF0b(j)=ctemp0
            iH0o=GF0b(j)
            GF1b(j)=ctemp1
            iH1o=GF1b(j)
          endif
        else
          if(i .eq. ip) then
            iH0o=GF0a(abs(j-i)+1)
            iH1o=GF1a(abs(j-i)+1)
          else
            iH0o=GF0b(abs(j-i)+1)
            iH1o=GF1b(abs(j-i)+1)
          endif
        endif
      endif
      LH0=H0+LH0
      if(Epol) then
        LH1=k1*m(j,2)/R*H1+LH1
        if(i .le. gN .and. j .le. gN) then
          LH1o=-LH1o-iH1o
          LH0o=iH0o+LH0o
        endif
      else
        LH1=k1*(nj-ni)/R*H1+LH1
        LH0o=(H0o-iH0o)+LH0o
      endif
204      continue
c      Coordinate rotation for source segment midpoints
      si=m(i,1)*cos(psi(1))+m(i,2)*sin(psi(1))
      ni=m(i,1)*sin(psi(1))-m(i,2)*cos(psi(1))
      if(neg(l)) then
        si=-si
        ni=-ni
      endif
      stepS=abs(stepS-si)
      DelS=2*stepS
c      HANKEL FUNCTION evaluation at midpoint of segment
      R=sqrt((sj-si)**2+(nj-ni)**2)
      if(Lossy) then
        ckrho=k1*R
        call cHank(ckrho,2,H0,H1)
      else
        krho=Real(k1)*R

```

```

        call Hankz1(krho,2,H0,H1)
    endif
    if(i .le. gN .and. j .le. gN) then
        krho=k*R
        call Hankz1(krho,2,H0o,H1o)
        if(i .eq. 1 .and. iter .eq. 1) then
            call impGF(krho,etab,Epol,ctemp0,ctemp1)
            GF0(j)=ctemp0
            iH0o=GF0(j)
            GF1(j)=ctemp1
            iH1o=GF1(j)
        else
            iH0o=GF0(abs(j-1)+1)
            iH1o=GF1(abs(j-1)+1)
        endif
    endif
c    GREEN'S FUNCTION INTEGRALS
    LH0=stepS/3*(4*H0+LH0)
    if(Epol) then
        LH1=stepS/3*(4*k1*m(j,2)/R*H1+LH1)
        if(i .le. gN .and. j .le. gN) then
            LH0o=stepS/3*(4*iH0o+LH0o)
            LH1o=LH0o+LH1o
        endif
    else
        LH1=stepS/3*(4*k1*(nj-n1)/R*H1+LH1)
        if(i .le. gN .and. j .le. gN) then
            LH0o=stepS/3*(4*(H0o-iH0o)+LH0o)
        endif
    endif
    if(i .ne. j .and. Rm .le. adj) then
        LH0=-LH0
    endif

    if(Epol) then
c    E-POL IMPEDANCE MATRIX
        Z(j,i)=k*Zo*ur/2*LH0
        if(i .le. gN .and. j .ne. i) then
            Z(j,N+i)=-ci/2*LH1
        else if(i .le. gN .and. j .eq. i) then
            Z(j,N+i)=-1.
        endif

        if(j .le. gN) then
            if(j .ne. i) then
                if(i .le. gN) then
                    Z(N+j,i)=-k/2*etab*LH1o
                else
                    Z(N+j,i)=czero
                endif
            else
                Z(N+j,i)=-1.-k/2*etab*LH1o
            endif
            if(i .le. gN) then
                Z(N+j,N+i)=-k*Yo/2*LH1o
            endif
        else
c    H-POL IMPEDANCE MATRIX
            if(j .ne. i) then
                Z(j,i)=ci/2*LH1
            else
                Z(j,i)=-1.
            endif
            if(i .le. gN) then
                Z(j,N+i)=k*Yo*er/2*LH0
            endif

            if(j .le. gN) then
                if(j .ne. i) then
                    if(i .le. gN) then
                        Z(N+j,i)=-k*etab/2*LH0o
                    else
                        Z(N+j,i)=czero
                    endif
                else
                    Z(N+j,i)=1.-k*etab/2*LH0o
                endif
            endif
            if(i .le. gN) then
                Z(N+j,N+i)=k*Yo/2*LH0o
            endif
        endif
    endif
210    continue

c...Incident Field (Source) matrix elements
    xj=m(i,1)
    Vj(i)=czero
    if(i .le. gN) then
        if(Epol) then
c    E-POL INCIDENT FIELD Hx
            Vj(N+i)=2*sin(phio)/(etab*sin(phio)+1)
            & *Yo*cexp(-ci*k*xj*cos(phio))
        else
c    H-POL INCIDENT FIELD Hz

```



```

      Vj(N+1)=2*sin(phio)/(etab+sin(phio))
      &      *cexp(-ci*k*xj*cos(phio))
    endif
  endif
220  continue
      istart=istop+1
      istop=istop+szN(1+1)
230  continue

c...Calling subroutines to calculate the current matrix
      call CGECO(Z,pn,NgN,ipvt,rc,wk)
      call CGESL(Z,pn,NgN,ipvt,Vj,0)

      print *, 'The condition number is ',rc
      do 310 i=1,NgN
c    CURRENT MATRIX
        Ii(i)=Vj(i)
310   continue

c***** Far Field Amplitude *****
c*****
      Psca=czero
      DelX=w/gN/2
      do 600 i=1,gN
c...Simpson's three point composite integration over each
c  segment in the aperture
        Lsca=DelX/3*(cexp(-ci*k*p(i,1)*cos(phi))
          &      +4*cexp(-ci*k*m(i,1)*cos(phi))
          &      +cexp(-ci*k*p(i+1,1)*cos(phi)))
        if(Epol)then
          Psca=(Ii(N+1)+etab*Zo*Ii(i))*Lsca+Psca
        else
          Psca=(Yo*Ii(N+1)-etab*Ii(i))*Lsca+Psca
        endif
600   continue
        if(Epol)then
          Psca=-k/2*sin(phi)/(etab+1)*Psca
        else
          Psca=-k/2/(etab+1)*Psca
        endif

        write(3,*) d,cabs(Psca)
        write(4,*) d,180/pi*(atan2(aImag(Psca),Real(Psca)))
        print *, 'Exact: |Psca| = ',cabs(Psca), ' arg Psca = ',
          & 180/pi*(atan2(aImag(Psca),Real(Psca)))
        d=d+dstep
700   continue

800   call exit
      END
c***** SUBROUTINE IMPGF (KRHO,ETAB,EPOL,LRE,LRM) *****
c*****
      real pi,krho,k,nu,nuo,numax
      real*8 rts(64,64),coef(64,64)
      complex ci,carg,croot,Ao
      complex re,rm,eint,mint,Lre,Lrm,H0,H1
      logical Epol,self,second
      common /data/ rts,coef
      call gaussq

      pi=3.141593
      ci=cmplx(0.,1.)
      k=2*pi
      gam=0.5772157
      Ao=1+ci*2/pi*(gam+alog(k/2))
      self=.false.
      second=.false.
      iroot=64
      ih=0

      alf=k/etab
      bet=k*etab
      nuo=krho/k
c...Determining size of del nu and max nu value necessary
      if(Epol)then
        dnu=alog(0.95)/(-alf)
        numax=alog(0.00001)/(-alf)
        ih=2
      else
        dnu=alog(0.95)/(-bet)
        numax=alog(0.00001)/(-bet)
      endif
      dnumax=sqrt((0.1/k)**2+nuo**2)-nuo
      if(dnu.gt. dnumax) dnu=dnumax
      rnumax=sqrt((12./k)**2+nuo**2)

      if(nuo .eq. 0.0)then
        self=.true.
        second=.true.
      endif

```

```

        if(dnu .ge. nuo .and. .not.(self))then
            dnu=nuo
            second=.true.
        endif
        if(numax .le. nuo+dnu) numax=2*(nuo+dnu)
        if(rnumax .lt. numax) numax=rnumax

c...Gauss-Quadrature
        Lre=0.0
        Lrm=0.0
        do 20 m=1,2
            if(.not.(m .eq. 1 .and. second))then
                if(m .eq. 1)then
                    a=0.0
                    b=nuo-dnu
                else if(m .eq. 2)then
                    a=nuo+dnu
                    b=numax
                else
                    a=b
                    b=numax
                endif

                mint=0.0
                eint=0.0
                do 70 i=1,iroot
                    srts=rts(iroot,i)
                    scoef=coef(iroot,i)
                    nu=(b-a)*srts+b+a)/2
                    croot=nuo**2-nu**2
                    carg=k*csqrt(croot)
                    if(m .eq. 1)then
                        rarg=Real(carg)
                        call Hankz1(rarg,ih,H0,H1)
                    else
                        call cHank(carg,ih,H0,H1)
                    endif
                    if(Epol)then
                        rm=alf*exp(-alf*nu)*nuo*H1/carg
                        re=alf*exp(-alf*nu)*H0
                    else
                        re=bet*exp(-bet*nu)*H0
                    endif
                    eint=scoef*re+eint
                    mint=scoef*rm+mint
70                continue

                Lre=(b-a)/2*eint+Lre
                Lrm=(b-a)/2*mint+Lrm
            endif
20        continue

c...Singularity evaluation, nuo = |xj-xi|
        if(.not.(self))then
            if(Epol)then
                mint=alf*exp(-alf*nuo)
                & * (nuo*dnu+ci/pi/k**2
                & * (alog((2*nuo-dnu)/(2*nuo+dnu))+ci*pi))
                eint=alf*exp(-alf*nuo)
                & * (2*ao*dnu
                & +ci/pi*((nuo+dnu)*(alog(2*dnu*nuo+dnu**2)
                & +ci*pi)-(nuo-dnu)*alog(2*dnu*nuo-dnu**2)
                & -4*dnu*nuo*(alog((2*nuo+dnu)/(2*nuo-dnu))
                & -ci*pi)))
            else
                eint=bet*exp(-bet*nuo)
                & * (2*ao*dnu
                & +ci/pi*((nuo+dnu)*(alog(2*dnu*nuo+dnu**2)
                & +ci*pi)-(nuo-dnu)*alog(2*dnu*nuo-dnu**2)
                & -4*dnu*nuo*(alog((2*nuo+dnu)/(2*nuo-dnu))
                & -ci*pi)))
            endif
            Lre=Lre+eint
            Lrm=Lrm+mint
            return
        end
C*****
SUBROUTINE SIMPGF(KRHO,ETAB,EPOL,LRE,LRM)
C*****
        real pi,k,nu,nuo,numax,umin,krho
        real*8 rts(64,64),coef(64,64)
        complex ci,carg,croot,Ao,H0,H1,H(64)
        complex re,rm,eint,mint,Lre,Lrm,ssLre,ssLrm
        logical Epol,self,second
        common /data/ rts,coef
        call gausq

        pi=3.141593
        ci=cplx(0.,1.)
        k=2*pi
        gam=0.5772157
        Ao=1+ci*2/pi*(gam+alog(k/2))
        self=.false.
        second=.false.
        iroot=64

```

```

numm=21
ih=0

alf=k/etab
bet=k*etab
nuo=krho/k
c...Determining size of del nu and max nu value necessary
if(Epol)then
  dnu=log(0.95)/(-alf)
  numax=log(0.00001)/(-alf)
  ih=2
else
  dnu=log(0.95)/(-bet)
  numax=log(0.00001)/(-bet)
endif
dnumax=sqrt((0.1/k)**2+nuo**2)-nuo
if(dnu .gt. dnumax) dnu=dnumax
rnumax=sqrt((12./k)**2+nuo**2)

if(nuo .eq. 0.0)then
  self=.true.
  second=.true.
  dx=real(Lrm)
  if(dnu .ge. dx) dnu=1.05*dx/2
  numin=dnu
endif
if(dnu .ge. nuo .and. .not.(self))then
  dnu=nuo
  second=.true.
endif
if(numax .le. nuo+dnu) numax=2*(nuo+dnu)
if(rnumax .lt. numax) numax=rnumax

c...Self-cell evaluation
if(self)then
  if(Epol)then
    mint=alf*exp(-alf*dnu/2)*2*(-dx/2*dnu+ci/pi/k**2
&      *alog((dx+dnu)/(dx-dnu)))
    eint=alf*exp(-alf*dnu/2)*2*((Ao-ci/pi**2)*dx*dnu
&      +ci/pi*dx*(dnu*alog(dx**2-dnu**2)-2*dnu
&      +dx*alog((dx+dnu)/(dx-dnu)))
&      +ci/pi/2*((dx**2-dnu**2)*(alog((dx-dnu)/(dx+dnu))
&      -ci*pi)+dx*dnu+ci*dx**2*pi))
    else
    eint=bet*exp(-bet*dnu/2)*2*((Ao-ci/pi**2)*dx*dnu
&      +ci/pi*dx*(dnu*alog(dx**2-dnu**2)-2*dnu
&      +dx*alog((dx+dnu)/(dx-dnu)))
&      +ci/pi/2*((dx**2-dnu**2)*(alog((dx-dnu)/(dx+dnu))
&      -ci*pi)+dx*dnu+ci*dx**2*pi))
    endif
    sslre=eint
    sslrm=mint
  endif

  rt=0.0
  13 if(rt .eq. 1.3) numm=numm+2

c...Integration over self-cell
do 30 mm=1,numm
  if(mm .gt. 1)then
    nuo=nuo+dx/(numm-1)
    if(abs(numin-nuo) .le. 0.000001)then
      rt=1.3
      nuo=0
      print *, 'ACK!!!!!! Changing numm.'
      goto 13
    endif
    if(Epol)then
      dnu=log(0.95)/(-alf)
    else
      dnu=log(0.95)/(-bet)
    endif
    dnumax=sqrt((0.1/k)**2+nuo**2)-nuo
    if(dnu .gt. dnumax) dnu=dnumax
    if(nuo .lt. numin)then
      second=.true.
      self=.true.
    else
      if(dnu .ge. (nuo-numin))then
        dnu=nuo-numin
        second=.true.
      endif
    endif
    if(numax .le. nuo+dnu) numax=2*(nuo+dnu)
    if(rnumax .lt. numax) numax=rnumax
  endif

c Gauss-Quadrature
  Lre=0.0
  Lrm=0.0
  do 20 m=1,2
    if(.not.(m .eq. 1 .and. second))then
      if(m .eq. 1)then
        a=numin
        b=nuo-dnu
      else if(m .eq. 2)then

```

```

        a=nuo+dnu
        if(nuo .lt. numin) a=numin
        b=numax
    else
        a=numax
        b=2*numax
    endif

    mint=0.0
    eint=0.0
    do 70 i=1,iroot
        srt=rt(iroot,i)
        scoef=coef(iroot,i)
        nu=(b-a)*srt+b+a)/2
        croot=((nu)**2-nu**2)
        carg=k*csqrt(croot)
        if(m .eq. 1) then
            rarg=real(carg)
            call Hankz1(rarg,ih,H0,H1)
        else
            call cHank(carg,ih,H0,H1)
        endif
        if(Epol) then
            rm=alf*exp(-alf*nu)*2*nuo*H1/carg
            re=alf*exp(-alf*nu)*H0
        else
            re=bet*exp(-bet*nu)*H0
        endif
        eint=scoef*re+eint
        mint=scoef*rm+mint
70    continue
        Lre=(b-a)/2*eint+Lre
        Lrm=(b-a)/2*mint+Lrm
    endif
20    continue
c...Singularity evaluation, nuo = |x-x1|
    if(.not.(self)) then
        if(Epol) then
            mint=alf*exp(-alf*nuo)
            & *2*(nuo*dnu+ci/pi/k**2
            & *(alog((2*nuo-dnu)/(2*nuo+dnu))+ci*pi))
            eint=alf*exp(-alf*nuo)
            & *(2*Ao*dnu
            & +ci/pi*((nuo+dnu)*(alog(2*dnu*nuo+dnu**2)
            & +ci*pi)-(nuo-dnu)*alog(2*dnu*nuo-dnu**2)
            & -4*dnu+nuo*(alog((2*nuo+dnu)/(2*nuo-dnu))
            & -ci*pi)))
        else
            eint=bet*exp(-bet*nuo)
            & *(2*Ao*dnu
            & +ci/pi*((nuo+dnu)*(alog(2*dnu*nuo+dnu**2)
            & +ci*pi)-(nuo-dnu)*alog(2*dnu*nuo-dnu**2)
            & -4*dnu+nuo*(alog((2*nuo+dnu)/(2*nuo-dnu))
            & -ci*pi)))
        endif
        Lre=Lre+eint
        Lrm=Lrm+mint
    endif

    H(mm)=Lre
    self=.false.
    second=.false.
30    continue
        Lre=0.0
        do 40 i=1,numm-2,2
            Lre=H(i)+4*H(i+1)+H(i+2)+Lre
40    continue
        Lre=ssLre+2*(dx/(numm-1))/3*Lre
        Lrm=ssLrm-Lrm
        return
    end
C*****
C SUBPROGRAM DATAINT CONTAINS INTEGRATION DATA
C*****
C
C     SUBROUTINE GAUSQ
C     REAL*8 RTS(64,64),COEF(64,64)
C
C     COMMON /DATA/ RTS,COEF
C
C-----+
C  FIXED POINTS FOR GAUSSIAN QUADRATURE  |
C-----+
C
C---N=64
C
RTS(64,1)= .999305041735772D0
RTS(64,2)= .996340116771955D0
RTS(64,3)= .991013371476744D0
RTS(64,4)= .983336253884625D0
RTS(64,5)= .973326827789910D0
RTS(64,6)= .961008799652053D0
RTS(64,7)= .946411374858402D0
RTS(64,8)= .929569172131939D0

```

```

RTS (64, 9) = .910522137078502D0
RTS (64, 10) = .889315445995114D0
RTS (64, 11) = .865999398154092D0
RTS (64, 12) = .840629296252580D0
RTS (64, 13) = .813265315122797D0
RTS (64, 14) = .783972358943341D0
RTS (64, 15) = .752819907260531D0
RTS (64, 16) = .719881850171610D0
RTS (64, 17) = .685236313054233D0
RTS (64, 18) = .648965471254657D0
RTS (64, 19) = .611155355172393D0
RTS (64, 20) = .571895646202634D0
RTS (64, 21) = .531279464019894D0
RTS (64, 22) = .489403145707052D0
RTS (64, 23) = .446366017253464D0
RTS (64, 24) = .402270157963991D0
RTS (64, 25) = .357220158337668D0
RTS (64, 26) = .311322871990210D0
RTS (64, 27) = .264687162208767D0
RTS (64, 28) = .217423643740007D0
RTS (64, 29) = .169644420423992D0
RTS (64, 30) = .121462819296120D0
RTS (64, 31) = .072993121787799D0
RTS (64, 32) = .024350292663424D0
RTS (64, 64) = -.999305041735772D0
RTS (64, 63) = -.996340116771955D0
RTS (64, 62) = -.991013371476744D0
RTS (64, 61) = -.983336253884625D0
RTS (64, 60) = -.973326827789910D0
RTS (64, 59) = -.961008799652053D0
RTS (64, 58) = -.946411374858402D0
RTS (64, 57) = -.929569172131939D0
RTS (64, 56) = -.910522137078502D0
RTS (64, 55) = -.889315445995114D0
RTS (64, 54) = -.865999398154092D0
RTS (64, 53) = -.840629296252580D0
RTS (64, 52) = -.813265315122797D0
RTS (64, 51) = -.783972358943341D0
RTS (64, 50) = -.752819907260531D0
RTS (64, 49) = -.719881850171610D0
RTS (64, 48) = -.685236313054233D0
RTS (64, 47) = -.648965471254657D0
RTS (64, 46) = -.611155355172393D0
RTS (64, 45) = -.571895646202634D0
RTS (64, 44) = -.531279464019894D0
RTS (64, 43) = -.489403145707052D0
RTS (64, 42) = -.446366017253464D0
RTS (64, 41) = -.402270157963991D0
RTS (64, 40) = -.357220158337668D0
RTS (64, 39) = -.311322871990210D0
RTS (64, 38) = -.264687162208767D0
RTS (64, 37) = -.217423643740007D0
RTS (64, 36) = -.169644420423992D0
RTS (64, 35) = -.121462819296120D0
RTS (64, 34) = -.072993121787799D0
RTS (64, 33) = -.024350292663424D0

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C  
C  
C

```

COEF (64, 1) = .001783280721696D0
COEF (64, 2) = .004147033260562D0
COEF (64, 3) = .006504457968978D0
COEF (64, 4) = .008846759826363D0
COEF (64, 5) = .011168139460131D0
COEF (64, 6) = .013463047896718D0
COEF (64, 7) = .015726030476024D0
COEF (64, 8) = .017951715775697D0
COEF (64, 9) = .020134823153530D0
COEF (64, 10) = .022270173808383D0
COEF (64, 11) = .024352702568710D0
COEF (64, 12) = .026377469715054D0
COEF (64, 13) = .028339672614259D0
COEF (64, 14) = .030234657072402D0
COEF (64, 15) = .032057928354851D0
COEF (64, 16) = .033805161837141D0
COEF (64, 17) = .035472213256882D0
COEF (64, 18) = .037055128540240D0
COEF (64, 19) = .038550153178615D0
COEF (64, 20) = .039953741132720D0
COEF (64, 21) = .041262563242623D0
COEF (64, 22) = .042473515123653D0
COEF (64, 23) = .043583724529323D0
COEF (64, 24) = .044590558163756D0
COEF (64, 25) = .045491627927418D0
COEF (64, 26) = .046284796581314D0
COEF (64, 27) = .046968182816210D0
COEF (64, 28) = .047540165714830D0
COEF (64, 29) = .047999388596458D0
COEF (64, 30) = .048344762234802D0
COEF (64, 31) = .048575467441503D0
COEF (64, 32) = .048690957009139D0
COEF (64, 64) = .001783280721696D0
COEF (64, 63) = .004147033260562D0
COEF (64, 62) = .006504457968978D0
COEF (64, 61) = .008846759826363D0
COEF (64, 60) = .011168139460131D0

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```

COEF (64,59) = .013463047896718D0
COEF (64,58) = .015726030476024D0
COEF (64,57) = .017951715775697D0
COEF (64,56) = .020134823153530D0
COEF (64,55) = .022270173808383D0
COEF (64,54) = .024352702568710D0
COEF (64,53) = .026377469715054D0
COEF (64,52) = .028339672614259D0
COEF (64,51) = .030234657072402D0
COEF (64,50) = .032057928354851D0
COEF (64,49) = .033805161837141D0
COEF (64,48) = .035472213256882D0
COEF (64,47) = .037055128540240D0
COEF (64,46) = .038550153178615D0
COEF (64,45) = .039953741132720D0
COEF (64,44) = .041262563242623D0
COEF (64,43) = .042473515123653D0
COEF (64,42) = .043583724529323D0
COEF (64,41) = .044590558163756D0
COEF (64,40) = .045491627927418D0
COEF (64,39) = .046284796581314D0
COEF (64,38) = .046968182816210D0
COEF (64,37) = .047540165714830D0
COEF (64,36) = .047999388596458D0
COEF (64,35) = .048344762234802D0
COEF (64,34) = .048575467441503D0
COEF (64,33) = .048690957009139D0

```

```

C
C---N=64
C

```

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RTS (64,1) = .999305041735772D0
RTS (64,2) = .996340116771955D0
RTS (64,3) = .991013371476744D0
RTS (64,4) = .983336253884625D0
RTS (64,5) = .973326827789910D0
RTS (64,6) = .961008799652053D0
RTS (64,7) = .946411374858402D0
RTS (64,8) = .929569172131939D0
RTS (64,9) = .910522137078502D0
RTS (64,10) = .889315445995114D0
RTS (64,11) = .865999398154092D0
RTS (64,12) = .840629296252580D0
RTS (64,13) = .813265315122797D0
RTS (64,14) = .783972358943341D0
RTS (64,15) = .752819907260531D0
RTS (64,16) = .719881850171610D0
RTS (64,17) = .685236313054233D0
RTS (64,18) = .648965471254657D0
RTS (64,19) = .611155355172393D0
RTS (64,20) = .571895646202634D0
RTS (64,21) = .531279464019894D0
RTS (64,22) = .489403145707052D0
RTS (64,23) = .446366017253464D0
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RTS (64,25) = .357220158337668D0
RTS (64,26) = .311322871990210D0
RTS (64,27) = .264687162208767D0
RTS (64,28) = .217423643740007D0
RTS (64,29) = .169644420423992D0
RTS (64,30) = .121462819296120D0
RTS (64,31) = .072993121787799D0
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RTS (64,64) = -.999305041735772D0
RTS (64,63) = -.996340116771955D0
RTS (64,62) = -.991013371476744D0
RTS (64,61) = -.983336253884625D0
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RTS (64,53) = -.840629296252580D0
RTS (64,52) = -.813265315122797D0
RTS (64,51) = -.783972358943341D0
RTS (64,50) = -.752819907260531D0
RTS (64,49) = -.719881850171610D0
RTS (64,48) = -.685236313054233D0
RTS (64,47) = -.648965471254657D0
RTS (64,46) = -.611155355172393D0
RTS (64,45) = -.571895646202634D0
RTS (64,44) = -.531279464019894D0
RTS (64,43) = -.489403145707052D0
RTS (64,42) = -.446366017253464D0
RTS (64,41) = -.402270157963991D0
RTS (64,40) = -.357220158337668D0
RTS (64,39) = -.311322871990210D0
RTS (64,38) = -.264687162208767D0
RTS (64,37) = -.217423643740007D0
RTS (64,36) = -.169644420423992D0
RTS (64,35) = -.121462819296120D0
RTS (64,34) = -.072993121787799D0
RTS (64,33) = -.024350292663424D0

```

```

C
C

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C

```
COEF(64,1)= .001783280721696D0
COEF(64,2)= .004147033260562D0
COEF(64,3)= .006504457968978D0
COEF(64,4)= .008846759826363D0
COEF(64,5)= .011168139460131D0
COEF(64,6)= .013463047896718D0
COEF(64,7)= .015726030476024D0
COEF(64,8)= .017951715775697D0
COEF(64,9)= .020134823153530D0
COEF(64,10)= .022270173808383D0
COEF(64,11)= .024352702568710D0
COEF(64,12)= .026377469715054D0
COEF(64,13)= .028339672614259D0
COEF(64,14)= .030234657072402D0
COEF(64,15)= .032057928354851D0
COEF(64,16)= .033805161837141D0
COEF(64,17)= .035472213256882D0
COEF(64,18)= .037055128540240D0
COEF(64,19)= .038550153178615D0
COEF(64,20)= .039953741132720D0
COEF(64,21)= .041262563242623D0
COEF(64,22)= .042473515123653D0
COEF(64,23)= .043583724529323D0
COEF(64,24)= .044590558163756D0
COEF(64,25)= .045491627927418D0
COEF(64,26)= .046284796581314D0
COEF(64,27)= .046968182816210D0
COEF(64,28)= .047540165714830D0
COEF(64,29)= .047999388596458D0
COEF(64,30)= .048344762234802D0
COEF(64,31)= .048575467441503D0
COEF(64,32)= .048690957009139D0
COEF(64,64)= .001783280721696D0
COEF(64,63)= .004147033260562D0
COEF(64,62)= .006504457968978D0
COEF(64,61)= .008846759826363D0
COEF(64,60)= .011168139460131D0
COEF(64,59)= .013463047896718D0
COEF(64,58)= .015726030476024D0
COEF(64,57)= .017951715775697D0
COEF(64,56)= .020134823153530D0
COEF(64,55)= .022270173808383D0
COEF(64,54)= .024352702568710D0
COEF(64,53)= .026377469715054D0
COEF(64,52)= .028339672614259D0
COEF(64,51)= .030234657072402D0
COEF(64,50)= .032057928354851D0
COEF(64,49)= .033805161837141D0
COEF(64,48)= .035472213256882D0
COEF(64,47)= .037055128540240D0
COEF(64,46)= .038550153178615D0
COEF(64,45)= .039953741132720D0
COEF(64,44)= .041262563242623D0
COEF(64,43)= .042473515123653D0
COEF(64,42)= .043583724529323D0
COEF(64,41)= .044590558163756D0
COEF(64,40)= .045491627927418D0
COEF(64,39)= .046284796581314D0
COEF(64,38)= .046968182816210D0
COEF(64,37)= .047540165714830D0
COEF(64,36)= .047999388596458D0
COEF(64,35)= .048344762234802D0
COEF(64,34)= .048575467441503D0
COEF(64,33)= .048690957009139D0
```

C

```
RETURN
END
```

## **APPENDIX C. Program Listing for the Quasi-Analytical Solution**

The quasi-analytical solution presented in Section 3 was programmed for solution, as listed in the program IMPQA.FTN below. The subroutines used by this program are listed in GAPSUB.FTN in Appendix A of [2].

For H-polarization, the far field amplitude  $P_H$  is calculated from (28), and for E-polarization,  $P_E$  is calculated from (34). The parameters  $a$  and  $b$  in (26) and (33), respectively, are dependent on the effective surface impedance  $\eta$  of the gap, which is calculated from the formulas in [2] according to the specified shape of the cavity and the field polarization.



```

C
C      IMPQA.FTN
C
C      This FORTRAN program computes the far field scattering due
C      to a narrow gap of specified shape in an infinite impedance
C      plane. The far field amplitude is calculated given the
C      input impedance of the gap, calculated from it's equivalent
C      transmission line model.
C
C      INPUT    The user is prompted from the subroutine
C               GAPROM for the polarization and angle of the
C               incident field, angle of far field observation,
C               relative permittivity of gap filling, shape and
C               dimensions of gap, segment size, number of
C               iterations with respect to gap depth, and
C               normalized input impedance.
C
C      OUTPUT FILES
C               GAPDAT  Contains input data.
C               IMPDAT  Effective surface impedance of the gap.
C               AMPDAT  Contains the magnitude of the far field.
C               PHADAT  Contains the phase of the far field.
C
C      SUBROUTINES
C               HANKZ1  Computes the Hankel functions of the first
C                       kind of orders zero and one.
C               CHANK   Computes the Hankel functions of the first
C                       kind of orders zero and one given a
C                       complex argument.
C               MODBES  Computes the modified Bessel functions of
C                       the first kind of orders zero and one.
C
C      FUNCTION
C               CTAN    Calculates the tangent of a complex
C                       argument.
C
C      integer EorH,N,noS,qN,szN(50)
C      real pi,k,phi,phio,w,d,maxC,q(50,2)
C      real dStp(50),wStp(50),I0,I1,krho,psi
C      complex czero,ci,ctemp,er,ur,kl,kc,carg,ctan,eta
C      complex Zi,Zl,Zc,ZL,X,B1,B2,H0,H1,a,b,Ke,Kh,Ao,Pasca,ge,gh
C      logical Epol,Lossy
C      common /prompt/ EorH,phio,phi,er,ur,igap,wStp,dStp,w,d,noS,
C      & q,maxC,noIter,etab
C
C      1  format(11)
C      2  format(15)
C      3  format(g16.8)
C      4  format(a1)
C      5  format(i3)
C      6  format(2g16.8)
C
C      open(1,file='gapdat')
C      open(2,file='impdat')
C      open(3,file='ampdat')
C      open(4,file='phadat')
C
C      c...Declaring constant values
C      czero=cmplx(0.0,0.0)
C      ci=cmplx(0.0,1.)
C      pi=4.0*atan(1.0)
C      k=2*pi
C      Eo=1.0
C      Ho=1.0
C      Zo=sqrt(4.e-07*pi/8.854e-12)
C      Yo=1./Zo
C      gam=0.5772157
C      iprg=2
C
C      c...Setting default values
C      10  EorH=1
C          phio=90.0
C          phi=90.0
C          er=cmplx(1.,0.0)
C          ur=cmplx(1.,0.0)
C          w=0.15
C          d=0.5
C          noS=3
C          maxC=0.01
C          noIter=30
C          etab=1.
C          adj=0.00001
C          Epol=.false.
C          Lossy=.false.
C          side=.true.
C
C      c...Prompting user for input data
C      15  call gaprom(iprg)
C
C          if(igap .eq. 5) GOTO 15
C          if(EorH .eq. 1) Epol=.true.
C          phio=phio*pi/180.0
C          phi=phi*pi/180.0
C          drat=dStp(1)/d

```



```

        Ri=sin(phio)/(sin(phio)*etab+1)
        Psca=-ci*pi/4*(k*w)**2*sin(phi)*Rsdp/ge*Ri*Ke
    else
        psi=1-exp(-1*(0.098+1.76*w)*etab)
        cw=0.245+1.267*w
        gh=exp(-cw*etab)*cexp(ci*psi)
        a=ci*2/k/w*Zo/(ETA-etab*Zo)/gh
        Kh=-1./(pi/2*a+0.1+log(2.))
        Ao=log(k*w/4)+gam-ci*pi/2
        Ri=sin(phio)/(etab+sin(phio))
        Psca=ci*pi*Rsdp/gh*Ri*Kh/(1+Ao*Kh)
    endif

c...Outputting the far field magnitude and phase
    print *, ' d = ', d
    print *, ' Analytical: |Psca| = ', cabs(Psca),
    & ' arg Psca = ', 180/pi*(atan2(aImag(Psca),Real(Psca)))
    write(3,*) d, cabs(Psca)
    write(4,*) d, 180/pi*(atan2(aImag(Psca),Real(Psca)))

    d=d+dstep
700    continue

    print *, ' Again (1=yes) ? '
    read(*,1) ians
    if(ians .eq. 1) GOTO 10

800    call exit
    END

```