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Singular Points of Reducible Sextic Curves

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There are 106 individual types of singular points for reducible complex sextic curves.

1. Introduction

Extensive studies of simple singularities of complex sextic curves have been made by Urabe [1, 2] and Yang [3]; see also [4]. Singular points of sextic curves of torus type have been classified by Oka and Pho [5, 6]. The authors have classified all individual types of singular points for irreducible real and complex sextic curves in a previous paper [7]. In this paper, we will determine the individual types of singular points for reducible complex sextic curves. There are 106 types. The proof is as elementary as possible and relies heavily on Puiseux expansions, which are computed by using Maple when necessary.

Definition of the Equivalence Relation. Let us begin with the definition of the equivalence relation on singular points of algebraic curves. The equivalence relation is that two singular points are equivalent if and only if their probranches have the same exponents of contact. The term "*probranch*" was used by Wall in [8] to refer to the distinct Puiseux expansions of an algebraic curve at a point. In the same book, Wall defines the *exponent of contact* between two probranches to be the smallest exponent such that the corresponding terms in the two Puiseux expansions have unequal coefficients. Given an algebraic curve with a singular point at the origin, let us now describe how to associate a tree diagram to this singular point once we have the Puiseux expansions. Each time at least one probranch separates, record the smallest exponent where that happens. Place all such exponents in a row at the top. For each exponent in the top row, there corresponds a column of vertices. Each Puiseux expansion corresponds to exactly one vertex in that column, and those expansions with

the same coefficients up to that exponent correspond to the same vertex. We start with one vertex on the left corresponding to the power zero. Line segments are drawn connecting the vertices from left to right, where each polygonal path from left to right corresponds to Puiseux expansions having the same set of coefficients up to a given exponent. The diagram stops at the first exponent where each vertex in that column corresponds to exactly one Puiseux expansion. This tree diagram uniquely specifies the singularity type (up to permutations of vertices within columns) provided that no tangent line at the origin is vertical. It follows from [8, Lemma 4.1.1] that the diagram we assign to a singular point is invariant under a linear change of coordinates. The diagram just codifies in the most convenient way all the information about the exponents of contact. For detailed elementary examples of this process for assigning diagrams to singular points, see the authors' papers [7, 9].

Classifications such as the one obtained in this paper can be applied to the construction of curves with controlled topology by a technique known as dissipation of singular points [10].

2. Examples of Proof Methods

Reducible sextic curves fall into families that must have an irreducible factor of degree one, two, or three. If there is an irreducible factor of degree one, then every type of singular point in that family can be obtained by careful scrutiny of the Newton polygon and a knowledge of all types of singular points for quintic curves. Next consider the cases where there is an irreducible factor of degree two or three (but none of degree one). If this factor does not share a common tangent line with the remaining factor(s), then the singular point types can be determined by mathematical common sense. If this factor does share a common tangent line with any of the remaining factor(s), then we study the family carefully by means of Maple computations.

All families of the latter type are displayed in the next section. In the final section of the paper, we give the complete list of 106 types of singular points for reducible complex sextic curves.

Conventions. We may assume that the singular point is at the origin. We do not consider curves with multiple components. We do not allow our singular points to have vertical tangent lines (change coordinates, if necessary).

We begin with some examples that illustrate what is meant by "careful scrutiny of the Newton polygon" when there is an irreducible factor of degree one. If the reducible sextic has a factor of degree one, it is necessary to consider each type of singular point for a quintic, and then consider what happens when a straight line through the origin is added. There are 41 cases to consider for the quintic curve (40 types of singular point of a complex quintic curve and the simple point). Let us consider three of these cases:

(1) the origin is a simple point of the quintic;

(2) the origin is an A_1 singular point of the quintic;

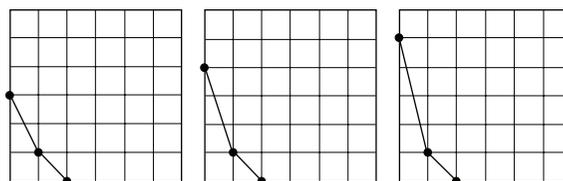(3) the origin is a D_6 singular point of the quintic.

(1) *Simple Point of the Quintic.* We may rotate axes so that the tangent line is $y = 0$. This means that the tangent cone is y. If the linear component is not $y = 0$, then we have an A_1 singular point. So now consider the case where the linear component is $y = 0$. We do not consider the case where y is a factor of the quintic curve because if the linear component is $y = 0$, then we have a multiple component. Since the quintic does not contain y as a factor, it

must contain either an x^2, or x^3, or x^4, or x^5 term (not an x term because $y = 0$ is tangent). The possible Newton polygons are
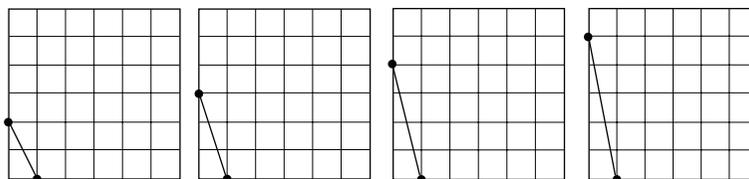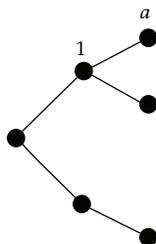

The corresponding singular points are A_3, A_5, A_7, A_9.

(2) *The Origin Is an A_1 Singular Point of the Quintic Component.* If the line component does not coincide with one of the two tangents of the A_1 singular point, then we get a D_4 singular point. Now change coordinates so that $y = 0$ is one of the tangent lines to the A_1 singular point and the other tangent line is not vertical. We now wish to add the line component $y = 0$. Thus, the Newton polygon for the quintic component is this

Then when we add the line component, the corresponding diagrams of the singular points for the resulting sextic are

with $a = 2, 3$, or 4. This gives a D_6, D_8, or D_{10} singular point, correspondingly.

(3) *The Origin Is a D_6 Singular Point of the Quintic Component.* If the line component does not coincide with one of the three tangents to the D_6 singular point, then we get an $X_{1,2}$ singular point. Diagram:
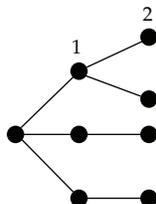
Suppose two probranches of the D_6 have tangent line T_1 and one probranch has tangent line T_2. Change coordinates so that T_1 becomes $y = 0$. Then the Newton polygon for the quintic is
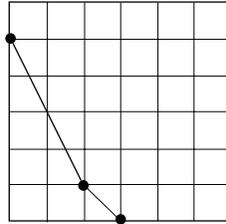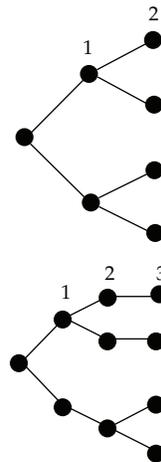

Then when we add the line component $y = 0$ (the exponent of contact with the two probranches of the quintic is 2), we get a singular point with the following diagram.
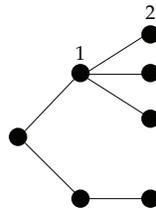
Next suppose we change coordinates so that T_2 becomes $y = 0$. Then the Newton polygons for the quintic are
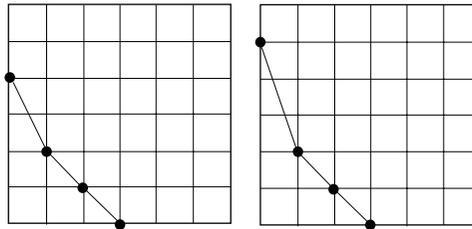
Then when we add the line component $y = 0$, we get singular points with the following diagrams.

Every other case with a line component can be completed with similar arguments.

Next consider the cases where there is an irreducible factor of degree two or three (but none of degree one). If this factor does not share a common tangent line with the remaining factor(s), then let us give examples of the use of "mathematical common sense." An irreducible factor of degree two has no singular point. So this case is like adding a line to a quartic, so nothing new is obtained (where the tangent line to the conic is not shared with any of the tangent lines of the quartic).

Now consider two irreducible cubics with a singular point. Each cubic has either an A_1 or an A_2 singular point or a simple point. Two A_1's give an X_9. Two A_2's give a $Y_{1,1}^1$. An A_1 and an A_2 give an $X_{1,1}$. An A_1 and a simple point give a D_4. An A_2 and a simple point give a D_5.

We now give examples of the proofs that relied on Maple computations. We will exhibit the derivation of all singular point types for two families where the components share at least one common tangent line. (We remind the reader that here we are not considering factors of degree one; this was settled above.)

Example 2.1. Consider the family $(y + ax^2 + bxy + cy^2)(y^2 + dxy + ex^3 + hy^3 + fx^2y + gxy^2 + jx^4 + kx^3y + lx^2y^2 + mxy^3 + ny^4 = 0)$, where $d \neq 0$.

By Maple computation, the Puiseux jets passing through the origin are $y = -dx$, $y = -(e/d)x^2$, $y = -ax^2$.

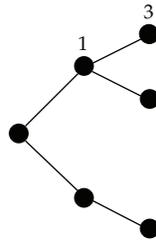If $a \neq e/d$, the diagram of the singular point is
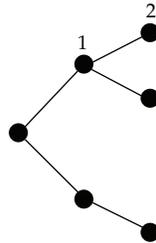
If $a = e/d$, this is substituted into the family and the Puiseux expansions are computed again. Note that $e \neq 0$; if $e = 0$, then the family would have a linear factor of y.

The Puiseux jets at the origin are now $y = -dx$, $y = -(e/d)x^2 + (b/e)x^3$, $y = -(e/d)x^2 + ((-e^2 - jd^2 + efd)/d^2)x^3$.

Set the two coefficients of x^3 equal to each other. Solving for j gives $j = -e(e - df + bd^2)/d^2$.

If $j \neq -e(e - df + bd^2)/d^2$, the diagram of the singular point is

If $j = -e(e - df + bd^2)/d^2$, this is substituted into the preceding family and the Puiseux expansion is computed.

The Puiseux jets at the origin are $y = -dx$, $y = -(e/d)x^2 + (be/d)x^3 + ((-e^2g + 2e^2b - ebdf + edk)/d^3)x^4$, $y = -(e/d)x^2 + (be/d)x^3 + (-eb^2d - e^2c)/d^2)x^4$.

Set the coefficients of x^4 equal to each other and solve for k: $k = (-b^2d^2 - cde + eg - 2be + bdf)/d$.

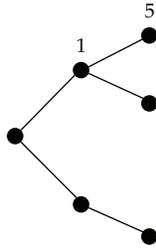If $k \neq (-b^2d^2 - cde + eg - 2be + bdf)/d$, then the diagram of the singular point is
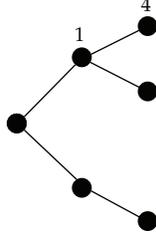
If $k = (-b^2d^2 - cde + eg - 2be + bdf)/d$, then substitute this into the preceding family and then compute the Puiseux expansion:

$$y = -dx,$$

$$y = -\frac{e}{d}x^2 + \frac{be}{d}x^3 + \frac{(-eb^2d - e^2c)}{d^2}x^4$$

$$+ \frac{(-2e^3c + e^3h - e^2b^2d + e^2cd^2b + e^2bdg + e^2cdf - e^2dl + d^3b^3e)}{d^4}x^5, \qquad (2.1)$$

$$y = -\frac{e}{d}x^2 + \frac{be}{d}x^3 + \frac{(-eb^2d - e^2c)}{d^2}x^4 + \frac{(eb^3d + 3e^2bc)}{d^2}x^5.$$

Set the coefficients of x^5 equal to each other and solve for l: $l = (-2cd^2b - 2ce + eh - b^2d + bdg + cdf)/d$.

If $l \neq (-2cd^2b - 2ce + eh - b^2d + bdg + cdf)/d$, then the diagram of the singular point is

If $l = (-2cd^2b - 2ce + eh - b^2d + bdg + cdf)/d$, then substitute this into the preceding family and compute the Puiseux expansion.

The Puiseux jets at the origin are $y = -dx$, $y = -(e/d)x^2 + (be/d)x^3 + ((-eb^2d - e^2c)/d^2)x^4 + ((eb^3d + 3e^2bc)/d^2)x^5 + ((-e^3dc^2 - e^3cg + 2e^3bc - e^3bh + e^3m - 6b^2d^2ce^2 - b^4d^3e)/d^4)x^6$, $y = -(e/d)x^2 + (be/d)x^3 + ((-eb^2d - e^2c)/d^2)x^4 + ((eb^3d + 3e^2bc)/d^2)x^5 + ((-6e^2db^2c - 2e^3c^2 - d^2b^4e)/d^3)x^6$.

Set the coefficients of x^6 equal to each other and solve for m : $m = -dc^2 - 2bc + bh + cg$.

If $m \neq -dc^2 - 2bc + bh + cg$, then the diagram of the singular point is

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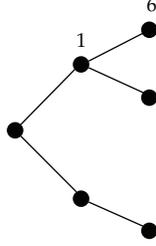
If $m = -dc^2 - 2bc + bh + cg$, then substitute into the preceding family and compute the Puiseux expansion.

The Puiseux jets at the origin are $y = -dx$, $y = -(e/d)x^2 + (be/d)x^3 + ((-eb^2d - e^2c)/d^2)x^4 + ((eb^3d + 3e^2bc)/d^2)x^5 + y = -dx$, $y = -(e/d)x^2 + (be/d)x^3 + ((-eb^2d - e^2c)/d^2)x^4 + ((eb^3d + 3e^2bc)/d^2)x^5 + ((-6e^2db^2c - 2e^3c^2 - d^2b^4e)/d^3)x^6 + ((-e^4c^2 + e^4ch - e^4n + 10c^2d^2be^3 + 10d^3b^3e^2c + d^4b^5e)/d^5)x^7$, $y = -dx$, $y = -(e/d)x^2 + (be/d)x^3 + ((-eb^2d - e^2c)/d^2)x^4 + ((eb^3d + 3e^2bc)/d^2)x^5 + ((-6e^2db^2c - 2e^3c^2 - d^2b^4e)/d^3)x^6 + ((b^5d^2e + 10cdb^3e^2 + 10c^2be^3)/d^3)x^7$.

Set the coefficients of x^7 equal to each other and solve for n: $n = -c^2 + ch$.

If $n \neq -c^2 + ch$, then the diagram of the singular point is

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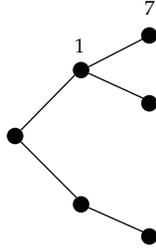
If $= -c^2 + ch$, then substitute this into the preceding family of curves. This time the Maple factor command tells us that the family has become

$$\frac{(dy + ex^2 + bdxy + cdy^2)^2(-ex^2 + dfx^2 - bd^2x^2 + dgxy + d^2x - d^2cxy - bdxy + dy - cdy^2 + dhy^2)}{d^3}.$$
(2.2)

The multiple factor tells us that the computation is done, and we have determined all singular point types belonging to the original family.

Example 2.2. Consider the family $(ax^3 + bx^2y + cxy^2 + dy^3 + y^2)(fx^3 + gx^2y + hxy^2 + jy^3 + y^2)$, where $a \neq 0$, and $f \neq 0$.

Maple computation gives the Puiseux jets at the origin:

$$y = \left(-\frac{x}{f}\right)^{3/2} f^2 \quad \text{(two expansions)},$$

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 \quad \text{(two expansions)}.$$
(2.3)

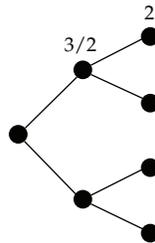If $a \neq f$, then the diagram of the singular point is
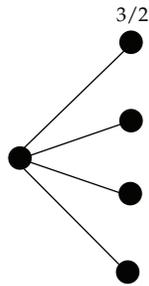
If $f = a$, substitute this into the family and recompute Puiseux expansions. The Puiseux jets at the origin are now

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{b}{2} x^2,$$

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{g}{2} x^2.$$

(2.4)
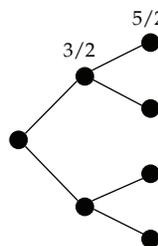
If $b \neq g$, then the diagram of the singular point is

If $b = g$, substitute this into the immediately preceding family and recompute Puiseux expansions. The Puiseux jets at the origin are

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{g}{2} x^2 + \left(-\frac{x}{a}\right)^{5/2} \left(\frac{1}{8} a^2 g^2 + \frac{1}{2} a^3 h\right),$$

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{g}{2} x^2 + \left(-\frac{x}{a}\right)^{5/2} \left(\frac{1}{2} ca^3 + \frac{1}{8} a^2 g^2\right).$$

(2.5)

If $c \neq h$, then the diagram of the singular point is
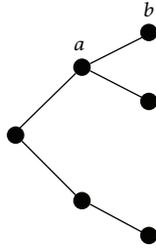
Figure 1: $a = 1$; $b = 2, 3, 4, 5, 6,$ or 7.

If $c = h$, then substitute this into the immediately preceding family and recompute Puiseux expansions. The Puiseux jets at the origin are

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{g}{2} x^2 + \left(-\frac{x}{a}\right)^{5/2} \left(\frac{1}{8} a^2 g^2 + \frac{1}{2} a^3 h\right) - x^3 \frac{\left(-j(a^4/2) - a^3 hg/2\right)}{a^3},$$

$$y = \left(-\frac{x}{a}\right)^{3/2} a^2 - \frac{g}{2} x^2 + \left(-\frac{x}{a}\right)^{5/2} \left(\frac{1}{8} a^2 g^2 + \frac{1}{2} a^3 h\right) - x^3 \frac{\left(-d(a^4/2) - a^3 hg/2\right)}{a^3}. \tag{2.6}$$

If $j \neq d$, then the diagram of the singular point is
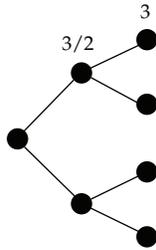
If $j = d$, then the family is $(ax^3 + gx^2 y + hxy^2 + dy^3 + y^2)^2$, and we are done; we have determined all singular points that come from the original family.

3. Results of Symbolic Computations

In this section we study the cases where the reducible sextic curve has an irreducible factor of degree two or three (but none of degree one) that shares a common tangent line with the remaining factor(s). Such families were studied carefully by means of Maple computations. Since these computations were quite lengthy, we refer the reader to the Maple worksheets posted on the website of Weinberg for the details [7]. Not every singular point of a sextic curve can be described by using the traditional Arnol'd notation. In this paper, we will express the singular point type by using the diagrams described in our previous papers [7, 9, 11]. For each family, we indicate the diagrams giving the singular point types.

(1) $(y+ax^2+bxy+cy^2)(y^2+dxy+ex^3+fx^2y+gxy^2+hy^3+jx^4+kx^3y+lx^2y^2+mxy^3+ny^4) = 0$ conic with one branch through origin, quartic with two distinct branches through origin, one tangent line in common (see Figure 1).
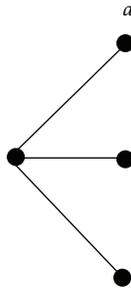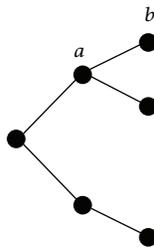
Figure 2: $a = 3/2$ or 2.

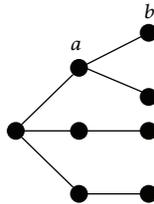**Figure 3:** $a = 2$; $b = 5/2, 3, 4, 5,$ or 6.

Figure 4: $a = 1$; $b = 2, 3, 4, 5,$ or 6.
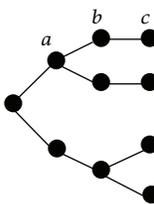
**Figure 5:** $a = 1$; $b = 3/2$; $c = 2$.

(2) $(y + ax^2 + bxy + cy^2)(y^2 + ex^3 + fx^2y + gxy^2 + hy^3 + jx^4 + kx^3y + lx^2y^2 + mxy^3 + ny^4) = 0$, conic with one branch through origin, quartic with 2 branches through the origin and one tangent line, the conic and the quartic sharing a common tangent line (see Figures 2 and 3).

(3) $(y + ax^2 + bxy + cy^2)(y^3 + dx^2y + exy^2 + fx^4 + gx^3y + hx^2y^2 + kxy^3 + ly^4) = 0$, conic with one branch through origin, quartic with three branches through origin, one tangent line in common (see Figures 4 and 5).

Figure 6: a = 2, 3, 4, 5, 6, 7, or 8.
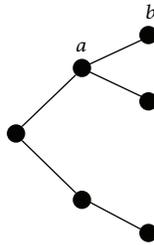
Figure 7: a = 2; b = 5/2, 3, 7/2, 4, 5, or 6.
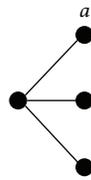
Figure 8: a = 2, 3, 4.
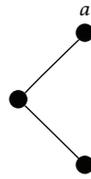
Figure 9: a = 2, 3, 4, 5, 6, 7, 8, or 9.
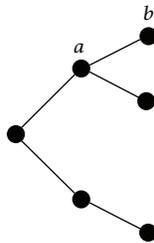
Figure 10: a = 1; b = 2, 3, 4, 5, 6, 7 or 8.

(4) $(y + ax^2 + bxy + cy^2)(y + dx^2 + exy + fy^2 + gx^3 + hx^2y + jxy^2 + ky^3 + lx^4 + mx^3y + nx^2y^2 + pxy^3 + qy^4) = 0$, conic with one branch through the origin, quartic with one branch through origin, one tangent line in common (see Figure 6).
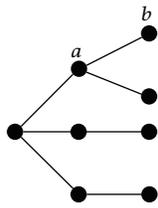
Figure 11: $a = 1$; $b = 2, 3, 4, 5,$ or 6.
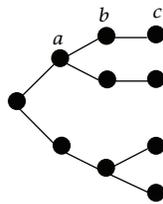
Figure 12: $a = 1$; $b = 2$; $c = 3, 4,$ or 5 and $a = 1$; $b = 3$; $c = 4$.
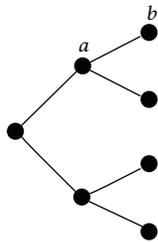
Figure 13: $a = 1$; $b = 2$ or 3.
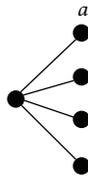
Figure 14: $a = 3/2$.
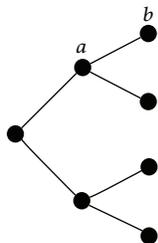
Figure 15: $a = 3/2$; $b = 2$ or $5/2$.

Figure 16: $a = 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2, 5, 11/2, 6, 13/2, 7, 8, 9, 10.$
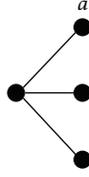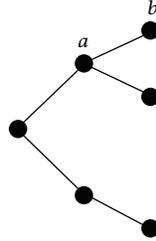

Figure 17: $a = 1, 4/3, 3/2, 5/3, 2, 3, 4.$

Figure 18: $a = 1; b = 3/2, 2, 5/2, 3, 7/2, 4, 9/2, 5, 11/2, 6, 13/2, 7, 8$ and $a = 2; b = 5/2, 3, 7/2, 4, 9/2, 5, 11/2, 6, 13/2$ and $a = 3; b = 4.$

(5) $(y+ax^2+bxy+cy^2)(y^2+dx^2y+ex^4+fxy^2+gy^3+hx^3y+jx^2y^2+kxy^3+ly^4) = 0$, conic with one branch through origin, quartic with two branches through origin, all three branches sharing the same tangent line (see Figures 7 and 8).

(6) $(y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + fxy^2 + gy^3)(y + hx^2 + jxy + ky^2 + lx^3 + mx^2y + nxy^2 + py^3)$, two cubics each with one branch through origin and a tangent line in common (see Figure 9).

(7) $(y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + fxy^2 + gy^3)(y^2 + hxy + jx^3 + kx^2y + lxy^2 + my^3)$, cubic with one branch through origin, cubic with two branches through origin, one tangent line in common (see Figure 10).

(8) $(y^2 + axy + bx^3 + cx^2y + dxy^2 + ey^3)(y^2 + fxy + gx^3 + hx^2y + jxy^2 + ky^3) = 0$, two cubics, each with two branches through origin, with one or two tangent lines in common (see Figures 11, 12, and 13).

(9) $(y^2 + bx^3 + cx^2y + dxy^2 + ey^3)(y^2 + gx^3 + hx^2y + jxy^2 + ky^3) = 0$, two cubics, each with two branches through origin, with a double tangent line in common (see Figures 14 and 15).

4. Summary of Singular Points of Reducible Sextic Curves
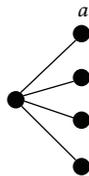
4.1. Multiplicity 2

See Figure 16.

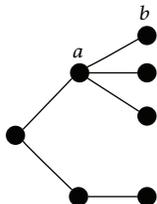**Figure 19:** $a = 1, 5/4, 4/3, 3/2$.

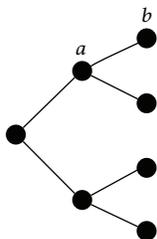**Figure 20:** $a = 1; b = 4/3, 3/2, 5/3, 2$.

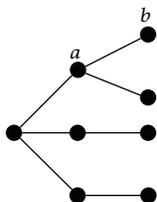**Figure 21:** $a = 1; b = 3/2, 2, 3$ and $a = 3/2; b = 2, 5/2$.

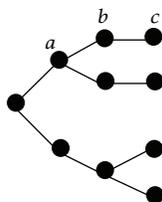**Figure 22:** $a = 1; b = 3/2, 2, 5/2, 3, 7/2, 4, 9/2, 5, 6$ and $a = 3/2; b = 2$.

Figure 23: $a = 1; b = 3/2; c = 2, 3, 4, 5, 6$ and $b = 2; c = 5/2, 3, 7/2, 4, 9/2, 5$ and $b = 5/2; c = 3$ and $b = 3$; $c = 7/2, 4, 9/2$.
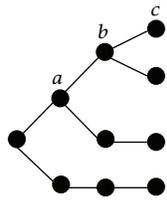
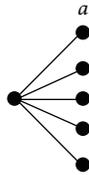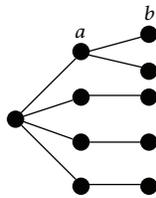**Figure 24:** $a = 1$; $b = 2$; $c = 5/2, 3, 7/2, 4, 9/2, 5$.

Figure 25: $a = 1, 5/4$.

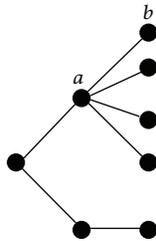**Figure 26:** $a = 1$; $b = 3/2, 2$.

Figure 27: $a = 1$; $b = 4/3, 5/4$.
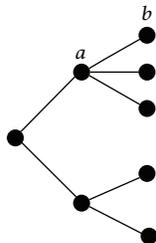
Figure 28: $a = 1$; $b = 3/2, 2$.
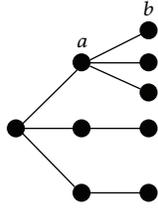
Figure 29: $a = 1$; $b = 4/3, 3/2, 2$.
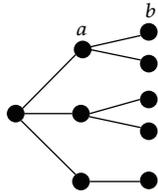
Figure 30: $a = 1$; $b = 3/2$.
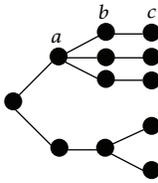
Figure 31: $a = 1$; $b = 4/3, 3/2$; $c = 2$.
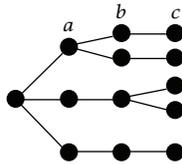

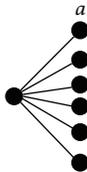**Figure 32:** $a = 1$; $b = 3/2$; $c = 2$.

Figure 33: $a = 1$.

4.2. Multiplicity 3

See Figures 17 and 18.

4.3. Multiplicity 4

See Figures 19, 20, 21, 22, 23, and 24.

4.4. Multiplicity 5

See Figures 25, 26, 27, 28, 29, 30, 31, and 32.

4.5. Multiplicity 6

See Figure 33.

Acknowledgment

References

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