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Modeling algae self-replenishment*

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Abstract

This paper presents a sunlight-dependent algae growth model. Driven by the circumstances surrounding Lake Chapala, Mexico, this theoretical model is an endeavor to understand the resilient sustainability of algae that threatens the area's ecosystem. In this paper, free-floating algae (phytoplankton) are treated as two distinct populations according to their location in the body of water: the vibrant sunlit upper region and the stagnate lower region where photosynthesis is not possible. The numerical solution for the model is analyzed and results are discussed in light of previous studies and the state of Lake Chapala.

Keywords : *Chapala, algae, phytoplankton, sunlight, model.*

1. Introduction

Lake Chapala, Mexico, known widely as a recreation destination, is home to numerous communities who make their living farming and fishing, and is the source of potable water for the city of Guadalajara's

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4 million people. Nestled in a mountain valley, this beautiful lake hosts a diverse throng of fish and migratory birds, and offers residents and tourists a comfortable climate with an average temperature about 20° C [3]. The Lerma Rivers system, which feeds into Lake Chapala and also provides water for Mexico City, is the ecological and hydrological core of central Mexico. As such, the crisis of Lake Chapala serves as an indicator of the health of the entire ecosystem, and its demise would be devastating to the economic development and ecology of the area.

The most visible ailment of Lake Chapala is dehydration. This loss of water is due to evaporation, drainage by the Santiago River to the Pacific Ocean, and piping of potable water to Guadalajara. The lake has come close to drying up completely a couple times in the last century and currently sits half-full with an average depth of 4.5 meters. In addition, the lake's once pristine waters are now muddy, with transparency rarely exceeding 30 centimeters. As a combined result of the dehydration and increased farming in the area, Lake Chapala has dangerously high levels of phosphorus and other sediments. Finally, the lake has been frequented by algal blooms, rapid increases in algae population, which have been irritations to the tourists and communities in the surrounding area [3].

Of the many factors that threaten Lake Chapala, we focus our attention on the algae, specifically referring to the free-floating phytoplankton. These algae are tiny drifting plants that grow and reproduce using photosynthesis. Although integral to a lake's ecosystem, some types of algae produce toxins, which can cause skin, ear, or eye irritation, or intestinal problems when swallowed or inhaled with water vapor in small amounts. Exposure in large amounts to these toxins can cause liver or nerve problems [10]. There was no evidence of algae in Lake Chapala until the citing of an algal bloom in 1994. Since then algal blooms have frequented the lake and the growth of seaweed, another form of algae, have hampered the use of motorized fisher boats [3]. Since the initial appearance of algae in 1994 in Lake Chapala, the blooms, though non-toxic, have at times tainted the smell and taste of potable water for the citizens of Guadalajara [3]. Currently no toxic strains of algae have been found in the lake, however, questions still linger concerning how the current strains of algae arrived and how the danger of toxic strains may be avoided [9].

A Citizens Report to the Commission for Environmental Cooperation and further investigation purport that the Mexican government is not

complying to protect the water quality and ecology of the Lerma Rivers system and is lax in their enforcement of the environmental laws concerning the region [5]. The Mexican government has defended its compliance with environmental law enforcement and its water management agency, the National Water Program, has defended its actions in restoring and preserving the water quality for the Lerma Rivers system [6]. Algae is just one factor in this environmental crisis, but is highly visible due to its effect on the taste and smell of drinking water in Guadalajara. Thus it has been at the forefront in media reporting of the crisis of Lake Chapala [3].

Considering the government efforts to eliminate the algae and the media reportings of algal blooms of increasing frequency and magnitude [9], the resilience of the algae in Lake Chapala cannot be neglected. Yet, is this resilience due to adaptability to chemicals used to kill them or do the algae survive for another reason, such as replenishment by water circulation? Compounding to the usual nutrients in the water that feed the algae, agricultural chemical residues, heavy metals, and dissolved solids in the lake have increased due to nitrations from upstream and to evaporation from the shallow lake [3]. These changes allow a large food base for the algae. Motorboats used in the lake increase water circulation and can extend the life of algae that sink to the lake bottom out of reach of the sun's light required for photosynthesis. This paper considers this circulated algae population as the core reason for the algae's resilience.

Previous models describing algae population dynamics fall into two categories: 'top-down' and 'bottom-up'. Top-down models focus on predator populations, such as zoo-plankton, as the controlling force of the algae population using a predator-prey coupling [13]. Bottom-up models [4, 8] focus on the dependencies of algae for growth, which include nutrients (nitrates and phosphates), sunlight and carbon dioxide, requirements for photosynthesis, and warm temperatures [12]. Several studies have collected data on algae populations at various lakes across the globe. One study done in Lake Kinneret, Israel from 1972 to 1993 includes data on algal biomass and chlorophyll [1]. Yet, research employing sunlight-dependent models is limited, as is research directly connecting algae blooms to Lake Chapala.

This paper is organized as follows: In section 2, a clear mathematical model for describing the sunlight-dependent algae population is developed. Analysis of the model is discussed in section 3, including phase-plane analysis and numerical solution. Section 4 describes the meaning

of the analytic results. Section 5 discusses the importance of this research and the path ahead.

2. Model description

In this section, we will explore the creation of a simple sunlight-dependent model to describe the resilient ability of an algae population to bounce back after near-extinction.

We start by splitting the water body, a vertical cross-section of the water, into two regions dependent upon the amount of sunlight received: upper, where photosynthesis is possible, and lower, where it is not. In these two sections reside the two algae population densities: u in the upper region, and v in the lower. Population densities, mass/volume, are used so that the model can be applied to any volume of water tested. Since algae depend on sunlight for photosynthesis, only the u population is capable of growth [12]. Also, these densities trade with one another through water circulation, a feasible assumption due to river inflow and watercraft churning. As the algae circulates, some is lost by sinking to the bottom, dying, or being eaten. Note that, though this simple model does not directly depend upon predators, the loss factor can be used to correct mild grievances in subsequent data. In essence, v is a perishable storage population that u circulates into and partially receives back from. This gives the following system:

$$\begin{aligned}\dot{u} &= \text{Growth} - \text{Circulation Out} + \text{Circulation In}, \\ \dot{v} &= \text{Circulation In} - \text{Circulation Out} - \text{Loss}.\end{aligned}$$

Assuming a sufficient amount of sunlight and nutrients, as is the case at Lake Chapala, a constant proportion of growth rate to population is used, befitting exponential growth. Yet, since this model is sunlight-dependent, the growth of algae in the upper region will be hierarchically inhibited as top layers of algae cast shadows on those below. Hence, the restricted growth can be modeled as $\frac{\alpha u}{1+\beta u}$, where α is the growth proportionality constant and β is the constant restriction rate. Though we are assuming no knowledge of the volume or cross-sectional area of the water, we do assume that they remain constant. Thus, ignoring loss, the circulation between the two regions, though not equal, will be a constant percentage of the current algae density in the region of origin: γu for the circulation downward and δv for the circulation upwards. Since the loss is also a percentage of the lower population, it can be combined with the upward

circulation as λv . The specified system is then

$$\begin{aligned} \dot{u} &= \frac{\alpha u}{1 + \beta u} - \gamma u + \delta v, \\ \dot{v} &= \gamma u - \lambda v, \end{aligned} \tag{1}$$

where all parameters are constant and nonnegative, and $\gamma, \delta, \lambda \leq 1$, since they are percentages. Rescaling the original system to nondimensional variables can greatly simplify the analysis of the model using these transformations:

$$x = \beta u, \quad y = \frac{\beta \lambda}{\gamma} v, \quad t' = \gamma t. \tag{2}$$

The system, now with only three parameters from the original five, becomes

$$\begin{aligned} \dot{x} &= \frac{Ax}{1+x} - x + By = f_1(x, y), \\ \dot{y} &= Cx - Cy = f_2(x, y), \end{aligned} \tag{3}$$

where $A = \alpha/\gamma$, $B = \delta/\lambda$, and $C = \lambda/\gamma$. Note, by construction, A, B, C are all nonnegative, and to circumvent trivial solutions, we will assume they are all positive.

3. Analysis

The model (3) is a nonlinear system of coupled *ordinary differential equations* (ODEs). When uncoupled, both ODEs are second order and classified as nonlinear, nonhomogeneous, and autonomous since we have a rational function in the first equation and neither depends explicitly on time [7]. By direct inspection of (3), we see that C dictates a limit of convergence in the second equation. Rewritten, $x = y + \frac{1}{C}\dot{y}$, so as C increases, y converges to x . Considering unscaled variables u and v , recall that $y = \beta \frac{\lambda}{\gamma} v = \beta C v$. So as C increases, v diminishes since y converges to x . Thus, this convergence dictates a system of only one algae population, which contradicts our assumption of a two population system and thus falls outside the focus of this paper. We will first analyze the system (3) by considering the associated nullclines and Jacobian matrix. With these, we will obtain the equilibrium points and investigate the criteria for their stability. The nullclines $\dot{x} = 0$ and $\dot{y} = 0$ separate positive and negative slope of the variables x and y . As solved for y they are $y = \frac{x^2 + (1-A)x}{B(1+x)}$, $y = x$. See Figure 1 for a demonstration of these nullclines and some plotted solutions.

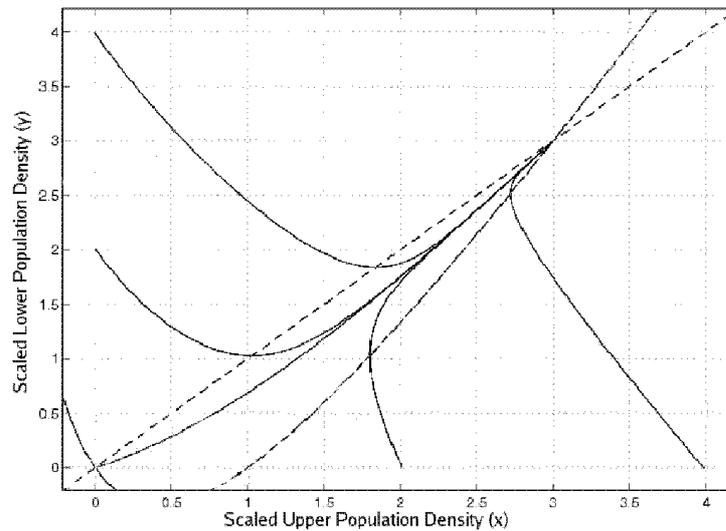


Figure 1

Phase Plane representation of the nullclines of the model (3) with $\alpha = 2$, $\gamma = \lambda = 1$, $\delta = \frac{1}{2}$ ($A = 2, B = \frac{1}{2}, C = 1$), and plotted solutions with various initial conditions (x_0, y_0)

The intersection points of these two nullclines,

$$(x_0, y_0) = (0, 0),$$

$$(x_*, y_*) = \left(\frac{A + B - 1}{1 - B}, \frac{A + B - 1}{1 - B} \right),$$

are the equilibrium points of the system: paired values at which both variables remain constant. Notice the parameter $C = \lambda/\gamma$ does not affect the nullclines or the equilibrium points. This, however, does not eliminate the effect of λ or γ , since B and A depend on them, respectively. To be biologically feasible, equilibria of algae population densities must be nonnegative. Nontrivially, this requires

$$\frac{A + B - 1}{1 - B} > 0.$$

Here the equality is dropped, for if it were not, the system would digress to one trivial equilibrium point, i.e. $(0, 0)$. To maintain this inequality, either the numerator and denominator are both positive or both negative. The latter case is impossible since it would assume a negative denominator, then $B > 1$, which forces $A + B > 1$, making it impossible to have a

negative numerator. Thus, both are positive, leading us to the feasibility criteria of the model:

$$\begin{aligned} A, B, C &> 0 \\ A + B &> 1 \\ B &< 1. \end{aligned} \tag{4}$$

The Jacobian matrix is the matrix of all first partial derivatives of the derivative functions, $\dot{x} = f_1$ and $\dot{y} = f_2$, from the system (3),

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix determine the behavior of the system. Using the determinant and trace of the Jacobian matrix with their combined discriminant, we can analyze the stability of the equilibrium points by using characteristics of the eigenvalues. This simplifies the task of equilibria characterization due to the complexity of the eigenvalues under three unknown parameters. The determinant, \det , and trace, Tr , are the product and sum of the eigenvalues of the matrix, respectively. In two dimensions, unstable saddle points are caused by eigenvalues with real parts of opposite sign, making the determinant of a two-dimensional system negative. Stable systems require eigenvalues to have negative real parts, making the trace negative. The discriminant, $\Delta = \text{Tr}^2 - 4 \det$, discriminates between real and complex eigenvalues; nonnegative for the former and negative for the latter [7].

At the origin, the Jacobian matrix is

$$\begin{pmatrix} A - 1 & B \\ C & -C \end{pmatrix}.$$

The determinant of this matrix is

$$\begin{aligned} \det &= (A - 1)(-C) - BC \\ &= -C(A + B - 1), \end{aligned}$$

which is negative by the feasibility criteria (4) as $A + B - 1 > 0$ and $A, B, C > 0$. Thus, as an equilibrium point, the origin is an unconditional saddle characterized by real eigenvalues of opposite signs.

Evaluated at the second equilibrium point $(x_*, y_*) = \left(\frac{A+B-1}{1-B}, \frac{A+B-1}{1-B}\right)$, the Jacobian matrix is

$$\begin{pmatrix} \frac{(1-B)^2}{A} - 1 & B \\ C & -C \end{pmatrix}.$$

The determinant of this matrix is

$$\det = \frac{BC(1-B)}{A},$$

and is positive by the feasibility criteria (4) as $1-B > 0$ and $A, B, C > 0$. So, (x_*, y_*) cannot be a saddle point. The trace of the Jacobian is

$$\begin{aligned} \text{Tr} &= \frac{(1-B)^2 - A(1+C)}{A} \\ &< \frac{(1-B)^2 - (1-B)(1+C)}{A} \\ &= -\frac{1-B}{A}(B+C), \end{aligned}$$

which is shown to be negative by first using the inequality $-A < -(1-B)$, and then using $1-B > 0$ and $A, B, C > 0$, all feasibility criteria (4). Since the determinant is positive and the trace is negative, (x_*, y_*) is a stable sink. To further describe the stability, we will look at the discriminant Δ :

$$\begin{aligned} &= \frac{(1-B)^2 - A(1+C)^2}{A} - 4 \left(\frac{BC(1-B)}{A} \right) \\ &= \frac{1}{A^2} [A^2((C+1)^2 - 4C + 4BC) + 2A(C-1)(1-B)^2 + (1-B)^4] \\ &= \left(\frac{A(C-1) + (1-B)^2}{A} \right)^2 + 4BC. \end{aligned}$$

Through simplification and completing the square, the discriminant is shown to be positive since $B, C > 0$. With a positive discriminant, no cyclic behavior is possible. Therefore, (x_*, y_*) is an unconditionally stable node characterized by two real, negative eigenvalues.

These results were confirmed graphically using the MATLAB software `pplane7` [11].

3.1 Phase-plane analysis

The phase plane provides a graphical interpretation of how the two population densities, x and y , vary against one another. The phase plane is used to determine stability and demonstrate qualitative tendencies of a system of two equations. Figure 2 demonstrates the stability of the

system (3), showing that all nontrivial solution curves arrive at the second equilibrium point, a nodal sink.

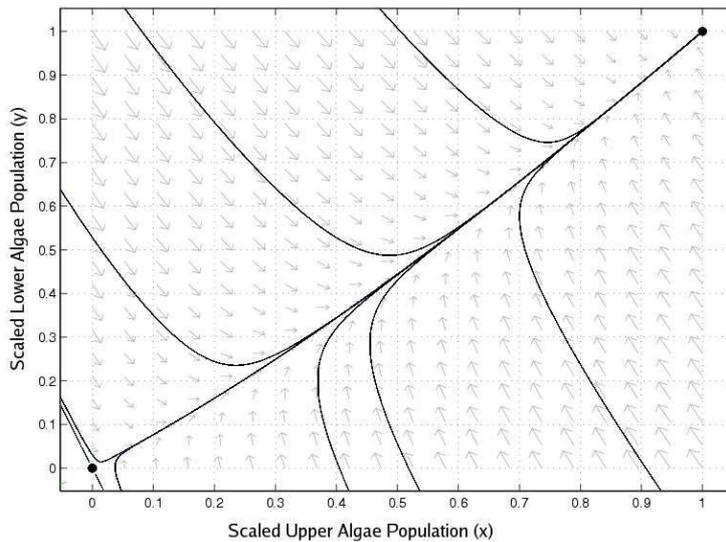


Figure 2

Phase Plane representation of solutions to the model (3) with $\alpha = 1, \gamma = \lambda = 1, \delta = \frac{1}{2}(A = 1, B = \frac{1}{2}, C = 1)$ and various initial conditions (x_0, y_0)

3.2 Numerical solution

No directly solvable form of the model (3) was found through transformations or standard approximations. A geometric approximation of the nonlinear term $\frac{Ax}{1+x}$, however, yielded $Ax(1 - x)$ when x is small ($x \ll 1$), making (3) similar to Verhulst’s differential equation known as the Logistic equation.

Using a fourth order Runge-Kutta method, a stable and consistent method, the numerical solution of the model (3) with chosen parameters is graphed through MATLAB for further analysis. Observations of Figure 3 show the model as a family of solutions similar to the logistic population model, Figure 4, where all nontrivial solutions converge to the nonzero equilibrium point [2]. The steepness of approach and time taken to reach equilibrium depend inversely on initial conditions and the parameter A . The y solution tracks the x solution with a certain amount of lag, dependent upon the value of C .

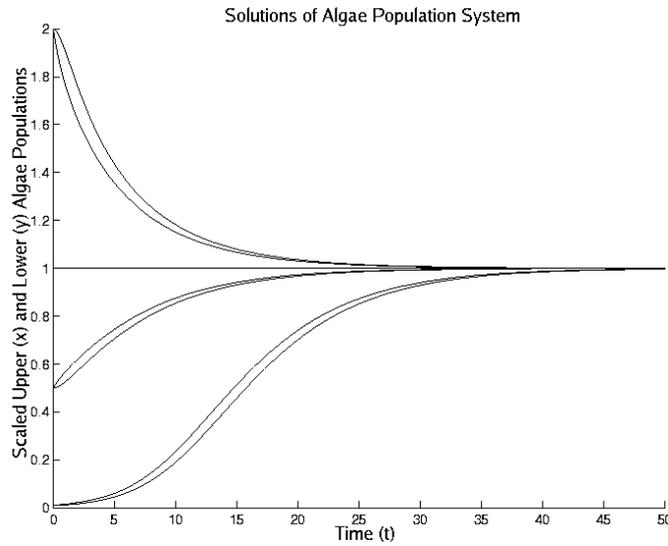


Figure 3

Phase plane representation of solutions to the model (3) with $\alpha = 1$, $\gamma = \lambda = 1$, $\delta = \frac{1}{2}$ ($A = 1, B = \frac{1}{2}, C = 1$) and various initial conditions (x_0, y_0)

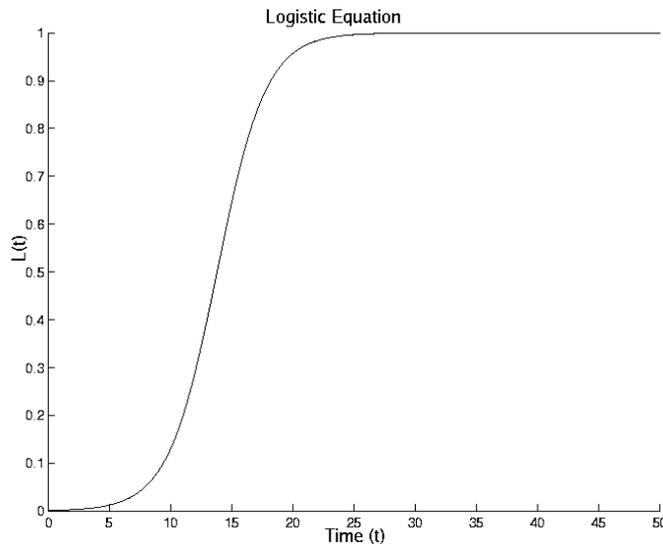


Figure 4

Phase plane representation of a solution to the model (3) with $\alpha = 1$, $\gamma = \lambda = 1$, $\delta = \frac{1}{2}$ ($A = 1, B = \frac{1}{2}, C = 1$) and the initial conditions $(x_0, y_0) = (0.001, 0.001)$

4. Results

Through applying the inequalities of the feasibility criteria to the Jacobian, the characteristics of the equilibrium points were determined. These results were confirmed using phase-plane analysis, and are summarized in Table 1. The phase plane and numerical solution confirmed the stability of the model (3), and also showed it to be robust, meaning that whatever feasible initial conditions are chosen, all solutions converge to the same value, $\frac{A+B-1}{1-B}$. Through analyzing the phase plane and the numerical solution, several affects of parameter changes on the solution were determined. Table 2 summarizes these results.

Table 1
Bounds on parameters

$\alpha, \beta > 0$	$0 < \gamma, \delta, \lambda \leq 1$
$\delta < \lambda$	$A, C > 0$
$0 < B < 1$	$A + B < 1$

Table 2
Effect of parameter changes

Parameter change	Effect
1. $C \rightarrow \infty$	1. $y \rightarrow x$ or $\frac{\lambda}{\gamma}v \rightarrow u$. Upper and lower population densities become proportional. Upward circulation is unevenly favored
2. $A + B \rightarrow 1$	2. Population densities increase more steeply. Either resulting from downward circulation greatly exceeding growth rate with a minimal loss, or downward circulation approaches growth rate and loss is great
3. $B \rightarrow 1$	3. Loss is minimized, and as a result the populations are quicker to replenish each other
4. $C \rightarrow 0$	4. Downward circulation is unevenly favored. Loss is minimal. Lower population becomes fixed

For equal scaled initial conditions x_0, y_0 , the true algae populations are initially proportioned by $u_0 = \frac{\lambda}{\gamma}v_0$. Under these conditions, the results clearly show that though the lower algae population acts like a decaying storage for the vibrant upper population, the lower population density, v , stays proportional to u , the upper population density, with a certain time lag.

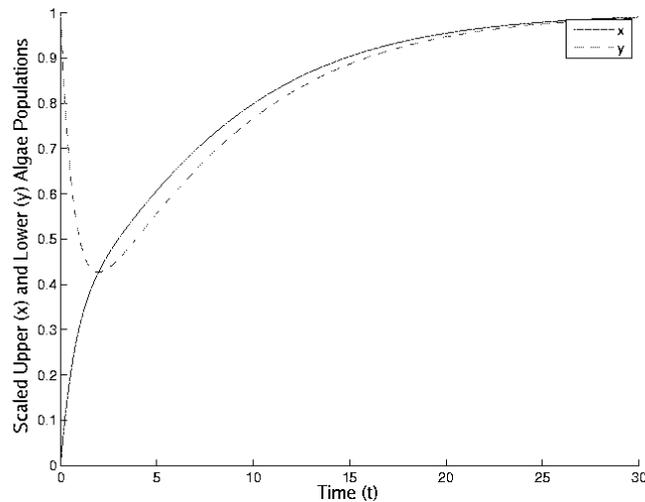


Figure 5

A numerical solution to the model (3) with $\alpha = 1$, $\gamma = \lambda = 1$, $\delta = \frac{1}{2}$ ($A = 1, B = \frac{1}{2}, C = 1$), and initial conditions $x_0 = 0$, $y_0 = 1$. This solution demonstrates through scaled variables how a decimated upper population of algae (x) can be revived through water circulation of the lower storage population (y)

Returning to the mystery surrounding the algae's resilience at Lake Chapala, the analysis supports two explanations of this phenomenon. One possibility is that the means used to destroy the algae had little or no effect. Therefore, the disappearance and reappearance of the algae depends on something else, like nutrient levels.

Yet there is another possibility. Following this model's separation of the algae population it is possible that those wishing to destroy the algae menace thoroughly destroyed the upper algae population but did not terminally harm the lower population that is deeper in the water. Figure 5 describes what happens in this case. As observed, the lower population supplies a leg-up in reviving the upper algae population. The mysterious reemergence could be simply the water redistributing the algae population through circulation.

5. Discussion

As the Lake Chapala ecosystem faces a crisis, this study proposes a theoretical analysis of this self-preserving algal phenomenon and presents

a model that focuses on a new facet of the ecological cycle of algae, namely sunlight, to view the impact of algae on the hydrological cycle of the Lerma Rivers system.

Many simulations and models have been developed to describe algal interactions in the aquatic food web and the dynamics of blooms and algal succession. Along with adding to this base, this study is proposed to spur further research on the algae of Lake Chapala that will hopefully guide governmental actions to preserve this vital ecosystem.

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