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Algae Population Self-Replenishment

R. Corban Harwood

George Fox University, rharwood@georgefox.edu

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STUDENT VERSION
Algae Population Self-Replenishment

Richard Corban Harwood

Department of Mathematics and Applied Sciences

George Fox University

Newberg OR USA

STATEMENT

Background Reading

Lake Chapala, Mexico, hosts a diverse throng of fish and migratory birds as well as numerous fishing and farming communities [1]. The Lerma Rivers system, the ecological and hydrological core of central Mexico, drains into Lake Chapala before draining out to the Pacific Ocean. As such, problems in Lake Chapala are indicators of the health of the entire ecosystem.

The most visible ailment of Lake Chapala is dehydration which has several side effects including algal blooms. This dehydration is due to a combination of evaporation, irrigation of it's headwaters to area farms, drainage by the Santiago River to the Pacific Ocean, and piping of potable water to Guadalajara's 4 million people. The lake has come close to drying up completely a couple of times in the last century and currently sits half-full with an average depth of 4.5 meters ($\approx$ 15 ft). In addition, the lake's once pristine waters are now muddy, with transparency rarely exceeding 30 centimeters ($\approx$ 1 ft). As a combined result of the dehydration and increased farming in the area, Lake Chapala has high levels of phosphorus and other sediments. Finally, the lake has been frequented by algal blooms, rapid increases in algae population, which have been irritations to the tourists and communities in the surrounding area [1].

Of the many factors that threaten Lake Chapala, we focus our attention on the algae, specifically referring to the free-floating phytoplankton. These algae are tiny drifting plants that grow and reproduce using photosynthesis. The first evidence of algal blooms in Lake Chapala was in 1994. Since then algal blooms have freqented the lake and the growth of seaweed, another form of algae, has hampered the use of motorized fisher boats [1]. Since the initial appearance of algae in 1994 in Lake Chapala, the blooms, though non-toxic, have at times tainted the smell and taste of potable

water for the citizens of Guadalajara [1]. Currently no toxic strains of algae have been found in the lake, however, questions still linger concerning how the current strains of algae arrived and how the danger of toxic strains may be avoided [5].

A Citizens Report to the Commission for Environmental Cooperation and further investigation purport that the Mexican government is not complying to protect the water quality and ecology of the Lerma Rivers system and is lax in their enforcement of the environmental laws concerning the region [2]. The Mexican goverment has assured environmental law enforcement with its compliance. Specifically, the National Water Program of Mexico has defended its actions in restoring and preserving the water quality for the Lerma Rivers system [3]. Algae is just one factor in this environmental contention, but is highly visible due to its effect on the taste and smell of drinking water in Guadalajara. Thus it has been at the forefront in media reportings of the crisis of Lake Chapala [1].

Considering the government efforts to eliminate the algae and the media reportings of algal blooms of increasing freqency and magnitude [5], the resilience of the algae in Lake Chapala cannot be neglected. Let us consider the possibility that algal blooms are being eliminated near the surface, but dormant algae deeper in the lake is unaffected. Churning in the water due to fishing motorboats brings this dormant population to the surface where they are activated by access to sunlight. The increase in agricultural chemical residues, heavy metals, and dissolved solids provide a rich food base for the algae to rapidly grow [1].

Reading questions:

1. What are some of the problems that algal blooms cause for the communities around Lake Chapala?

2. How does water depth impact a free-floating algae?

Modeling Assumptions

You will develop a sunlight-dependent model to describe the resilient ability of an algae population to bounce back after near-annihilation.

The modeling process starts with applying common simplifying assumptions, such as spatial uniformity. Assuming that the density of algae is uniform horizontally, we reduce population change to time and depth. Assuming a threshold depth, after which photosynthesis is no longer possible, we can discretize the depth variability into two regions or compartments with spatially uniform population densities. Thus, we are left with an upper population density of active algae $u(t)$ which are photosynthesizing and a lower dormant population $v(t)$ with units g/m^3 or other volumetric density units, splitting the lake into two domains represented by points where these population densities reside but still vary in time. Time, t, is usually measured in days to average out the daily cycle of sun intensity and focus on the scale of days to months for population growth. Volumetric population densities are used instead of population counts so that the model can be applied to any volume of water tested.

Since algae depend on sunlight for photosynthesis, only the u population is capable of growth [9]. Due to circulation in the water (based on river inflow and watercraft propulsion), these densities mix with one another. As the algae circulate with the water, some are lost by either sinking to the bottom, dying, or being eaten. Note, this loss term provides only a first order approximation of the effect of predation. Expanding the system to include a predator term may mitigate significant errors when comparing to subsequent data. In essence, v is a perishable storage population that u circulates into and partially receives back from. This gives the following system:

$$\frac{du}{dt} = \text{Growth - Circulation Down + Circulation Up,}$$
$$\frac{dv}{dt} = \text{Circulation Down - Circulation Up - Loss.}$$

Using PPLANE Software

A powerful tool for visualizing and interactively analyzing systems of differential equations is PPLANE, an executable Java applet exported from MATLAB source code which can be run without having MATLAB, but it does require Java Runtime Environment installed [8]. A short tutorial with example image follow below. For a more extensive tutorial, see [7]. PPLANE 2005.10 was updated in 2015 by Dr. Joel Castellanos at the University of New Mexico from code originally developed in 1994 by Dr. John Polking at Rice University. The file pplane.jar is available to download at http://math.rice.edu/~dfield/dfpp.html. Documentation and original MATLAB source code is also availabe at the site.

Figure 1 shows an example display of PPLANE with a system similar to model (1), but focusing solely on circulation between the two compartments:

$$\frac{du}{dt} = -au + bv,$$
$$\frac{dv}{dt} = au - bv.$$

Here is a quick tutorial for first time users of PPLANE. Download and open the PPLANE.jar file. The copyright page gives a brief summary of what the PPLANE software can do; click "Ok". In the PPLANE equation window, enter your system by typing the dependent variables u, v into the top left boxes and their derivative functions in the boxes to their right. Note that multiplied variables can be written sequentially with or without an '*' between them, unlike the syntax of the underlying MATLAB code. Parameter values ($a = 1/2$ and $b = 1/3$ for example) need to be defined in the boxes in the lower left and the window size dimensions (min $x = 0$, max $x = 1$, min $y = 0$, max $y = 1$) go in the lower right. Note, the 'x' and 'y' here refer to the first and second dependent variables. Clicking on "Graph Phase Plane" will produce the phase plane where you can click multiple times on the screen to see curves solved numerically using the clicked position as the initial position $u(0), v(0)$. To see the individual functions of time, select the "Graph" header, "both $x - t$ & $y - t$", and then click a point on a solution curve you have already plotted. To clean up
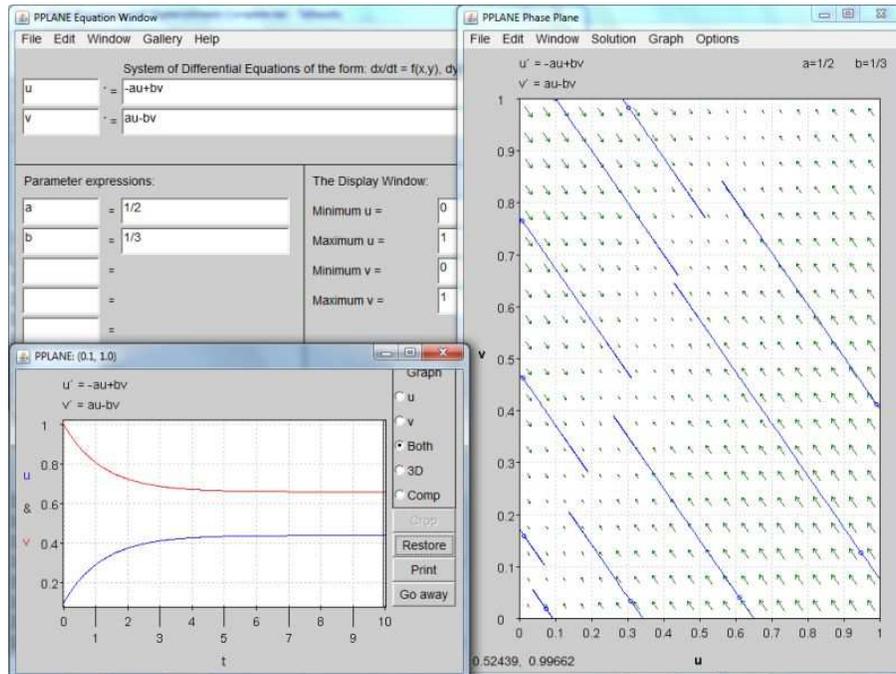
Figure 1. Phase plane representation of circulation between two compartments with $a = \frac{1}{2}, b = \frac{1}{3}$, with several plotted solution curves and a graph of an example function.

the graph of the individual function to a more reasonable interval, you can click the vertical bar following the cursor twice for lower and upper bound and click "Crop" to reduce the graph.

ACTIVITIES

1. **Set Up Model:** Fill in the verbal model with your own functions of u, v for each slope function $u' = f_1(u, v)$ and $v' = f_2(u, v)$. Start simple and explain your selections to someone else for feedback. Nondimensionalize your model by scaling and combining parameters to reduce the number of parameters in the model.

   Hint: One simple method is to scale everything by a common factor and make substitutions, such as turning $\frac{du}{dt} = bu(1 - au) + bv$ into $\frac{dx}{d\tau} = x(1 - x) + y$, where $x = au, y = av, \tau = bt$. Note, it may be helpful to rename all variables and parameters in a common way. Note, in this example, all parameters have been built into the new variables and thus will impact the solution behavior indirectly based upon bounds on x and y.

2. **Predict Behavior:** Type your model into PPLANE (see tutorial and Figure **??**) and choose parameter values that seem feasible to you based upon conditions on the original parameters (a good initial guess). Check the plot and describe the behavior. Are solutions converging

(stable or cyclic) or diverging (unstable)? Are solutions moving in exponential or straight paths (real eigenvalues) or spiraling (nonreal eigenvalues)? Alter the parameter values. The solution behaviors should shift, but does the type of behavior (different answers the the above questions) change?

3. **Linearize Model:** Using your nondimensionalized version of your model, find equilibirum points by setting your slope functions equal to zero and solving for values of the variables satisfying both equations. Determine the feasibility conditions which need to be placed upon your new variables and the original variables (reversing your substitutions) to have nonnegative equilibrium points. These represent conditions for biologically feasible equilibrium populations.

4. **Analyze Jacobian:** Then form the Jacobian matrix at each point and analyze the solution behavior based upon the trace and determinate of each Jacobian matrix.

   Hint: Using $\frac{dx}{d\tau} = f_1(x, y), \frac{dy}{d\tau} = f_2(x, y)$ as an example nondimensionalized system, the Jacobian matrix would be

   $$J(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

5. **Analyze Parameters:** Summarize feasibility (conditions for nonnegative equilibria) and stability conditions (conditions which make the nonnegative equilibria a stable node) on parameters. Use PPLANE to explore this parameter space and describe the amount of increasing/decreasing for each parameter upon the scaled and original variables. Demonstrate an example of each type of behavior for feasible equilibria.

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