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AN ELEMENTARY PROOF OF DODGSON'S CONDENSATION METHOD FOR CALCULATING DETERMINANTS

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ABSTRACT. In 1866, Charles Ludwidge Dodgson published a paper concerning a method for evaluating determinants called the condensation method. His paper documented a new method to calculate determinants that was based on Jacobi's Theorem. The condensation method is presented and proven here, and is demonstrated by a series of examples. The condensation method can be applied to a number of situations, including calculating eigenvalues, solving a system of linear equations, and even determining the different energy levels of a molecular system. The method is much more efficient than cofactor expansions, particularly for large matrices; for a 5×5 matrix, the condensation method requires about half as many calculations. Zeros appearing in the interior of a matrix can cause problems, but a way around the issue can usually be found. Overall, Dodgson's condensation method is an interesting and simple way to find determinants. This paper presents an elementary proof of Dodgson's method.

1. INTRODUCTION

Determinants are often used in linear algebra and other mathematical situations. They first were developed in the 18th century, and have since become ubiquitous in certain areas of mathematics. Although small determinants are quite simple to calculate, the degree of difficulty increases exponentially with the size of the matrix. An interesting way of finding determinants that can be much simpler than traditional methods came from an unlikely source. Reverend Charles Ludwidge Dodgson, better known by his pseudonym Lewis Carroll as the creator of *Alice in Wonderland*, was also a recreational mathematician [1]. He was especially fascinated by Euclid and published several works on Euclidean geometry, although he did publish some other papers and books on various topics, none of which became well known [1]. In the year 1866, he published a paper in the Proceedings of the Royal Society of London that first presented the condensation method to the public [1]. In his paper Dodgson outlined his new method for computing determinants, which he called "far shorter and simpler than any [method] hitherto employed"; he also proved the method for 3×3 and 4×4 matrices [2]. Relative to using a cofactor expansion (the traditional method), the condensation method can be quite a time-saver when dealing with larger matrices. Dodgson's based his method on Jacobi's Theorem, a "well-known theorem in determinants" [2]. The method can be utilized to improve calculation efficiency in many applications that involve finding determinants, such as solving systems of equation using Cramer's Rule, determining if a matrix is invertible, and finding the eigenvalues of a matrix [1][3]. Determinants also arise in other scientific fields; some examples are engineering, physics, and as we shall see, chemistry. Dodgson's method is thus applicable to a wide variety of situations and subjects. This paper will examine the theoretical basis of the condensation method, as well as presenting the method and some examples, and finally proving the method.

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2. THEORY

2.1. Definitions. The following definitions are relevant to the consideration of Charles Dodgson's condensation method, and will be necessary to understand the method and its applications.

Definition 1. For an $n \times n$ matrix, a *minor* is any $(n - m) \times (n - m)$ matrix formed by deleting m rows and m columns from the original matrix. For example, the italicized entries in the matrix below form a 2×2 minor in the upper left corner of the matrix.

$$\begin{pmatrix} \mathit{2} & \mathit{1} & -1 & -3 \\ \mathit{1} & \mathit{-2} & 3 & 0 \\ 3 & 1 & 2 & -1 \\ 0 & -2 & 3 & 1 \end{pmatrix}$$

Definition 2. A *consecutive minor* is a minor in which the remaining rows and columns were adjacent in the original matrix. Thus the example above shows a consecutive minor, as does the example below.

$$\begin{pmatrix} 2 & \mathit{1} & \mathit{-1} & -3 \\ \mathit{1} & \mathit{-2} & \mathit{3} & 0 \\ 3 & 1 & 2 & -1 \\ 0 & -2 & 3 & 1 \end{pmatrix}$$

Definition 3. A *complementary minor* is the $m \times m$ matrix diagonally adjacent to the minor matrix. The italicized entries below form a minor complementary to the original minor.

$$\begin{pmatrix} 2 & 1 & -1 & -3 \\ \mathit{1} & \mathit{-2} & \mathit{3} & 0 \\ 3 & \mathit{1} & \mathit{2} & -1 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

Definition 4. The *interior* of a matrix is the matrix formed by deleting the first and last rows and columns of the matrix. So for our example matrix the interior is the italicized entries shown below.

$$\begin{pmatrix} 2 & 1 & -1 & -3 \\ \mathit{1} & \mathit{-2} & \mathit{3} & 0 \\ 3 & \mathit{1} & \mathit{2} & -1 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

Definition 5. A *connected minor* is one in which all of the rows and columns are adjacent. All of the examples shown have been connected minors.

Definition 6. The *adjugate* matrix is defined entrywise by $a'_{ij} = (-1)^{i+j} \cdot \det[A_{ij}]$, where $[A_{ij}]$ is the minor with row i and column j deleted. It is denoted A' .

2.2. Jacobi's Theorem. The foundation of Dodgson's method can be found in Jacobi's Theorem, a result first stated by Jacobi in 1833, and then further developed in papers published in 1835 and 1841, by which time the theorem was fully realized [4].

Theorem. *Jacobi's Theorem.* Let A be an $n \times n$ matrix, let $[A_{ij}]$ be an $m \times m$ minor of A , where $m < n$, let $[A'_{ij}]$ be the corresponding $m \times m$ minor of A' , and let $[A^*_{ij}]$ be the complementary $(n - m) \times (n - m)$ minor of A . Then:

$$\det[A'_{ij}] = \det(A)^{m-1} \cdot \det[A^*_{ij}]$$

Dodgson noted that by considering the case where m is 2 and dividing by $\det[A^*_{ij}]$ a useful algorithm for finding the determinant of A could be obtained:

$$\det(A) = \frac{\det[A^*_{ij}]}{\det[A^*_{ij}]}$$

In their article about Charles Dodgson and his interesting method, Rice and Torrence outline a proof of Jacobi's Theorem [1]. Those interested can find it there.

2.3. Condensation Method. The condensation method developed by Charles Dodgson is as follows:

- First remove all zeros from the interior of A , using elementary row and column operations. Call this matrix $A^{(0)}$.
- Then take the determinants of the 2×2 consecutive minors to form an $(n-1) \times (n-1)$ matrix $A^{(1)}$.
- Next, take the determinants of the 2×2 consecutive minors in $A^{(1)}$ to form an $(n-2) \times (n-2)$ matrix. Divide each term by the corresponding entry from the interior of matrix $A^{(0)}$ to form $A^{(2)}$.
- In general, given $A^{(k)}$, compute an $(n-k-1) \times (n-k-1)$ matrix from the determinants of the 2×2 consecutive minors of $A^{(k)}$. To produce $A^{(k+1)}$ divide each entry by the corresponding entry in the interior of $A^{(k-1)}$.
- Repeat the previous step until a single number is obtained, which is $\det(A)$.

3. EXAMPLES AND APPLICATIONS

3.1. Examples. Now we shall consider some examples that illustrate the use of Dodgson's condensation method.

Example. Let

$$A = \begin{pmatrix} 4 & 2 & 0 & -3 \\ 1 & 1 & 2 & 2 \\ 0 & -1 & 3 & -1 \\ 1 & 2 & 5 & 1 \end{pmatrix}$$

Applying the condensation method, we obtain:

$$A^{(1)} = \left(\begin{array}{c|c|c} \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -3 \\ 2 & 2 \end{vmatrix} \\ \hline \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} \\ \hline \begin{vmatrix} 0 & -1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} \\ \hline \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 5 & 1 \end{vmatrix} \end{array} \right) = \begin{pmatrix} 2 & 4 & 6 \\ -1 & 5 & -8 \\ 1 & -11 & 8 \end{pmatrix}$$

So then

$$A^{(2)*} = \begin{pmatrix} 14 & -62 \\ 6 & -48 \end{pmatrix}$$

Dividing each entry by the corresponding entry of the interior of A - in other words, taking the upper left entry of $A^{(2)*}$, 14, and dividing by the upper left entry of the interior of A , 1, then continuing in this fashion, we get

$$A^{(2)} = \begin{pmatrix} 14 & -31 \\ -6 & -16 \end{pmatrix}$$

Thus, $A^{(3)*} = (-410)$, and dividing by the interior of $A^{(1)}$ (which is 5), we get the result $A^{(3)} = -82 = \det(A)$.

Example. Let us consider another example. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 4 \\ -1 & 3 & 6 & -3 \\ 5 & 1 & 2 & 0 \\ -2 & 1 & -1 & 1 \end{pmatrix}$$

Then we find

$$A^{(1)} = \begin{pmatrix} 1 & 6 & -24 \\ -16 & 0 & 6 \\ 7 & -3 & 2 \end{pmatrix}$$

This matrix has a zero in the interior, which will cause problems later on. We can use row operations to change A so we are able to find the determinant. We rearrange A by placing the first row at the bottom and moving all the other rows up one, for a total of 3 row interchanges. We now have the matrix

$$B = \begin{pmatrix} -1 & 3 & 6 & -3 \\ 5 & 1 & 2 & 0 \\ -2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 4 \end{pmatrix}$$

Now after we take determinants of consecutive minors, we get

$$B^{(1)} = \begin{pmatrix} -16 & 0 & 6 \\ 7 & -3 & 2 \\ -2 & 1 & -4 \end{pmatrix}$$

As you can see, the zero is no longer in the interior of the matrix, so we can proceed in our calculations. Continuing on, we eventually obtain $\det(B) = 163$. Since we initially performed 3 row interchanges, we must multiply our answer by $(-1)^3$, and we find that $\det(A) = -163$.

3.2. Applications. The condensation method can be applied to many situations where calculation of a determinant is needed. One such situation is in quantum mechanics, where determinants can be used to determine the energy levels of molecules (specifically π electron systems) using Molecular Orbital Theory. Using some experimental data, as well as theory developed for simple systems, we can generate what is called a system of secular equations that can be solved by setting the determinant of the system equal to zero. The solutions to this system give the energy levels of the particular molecule being studied; knowing the energy levels is crucial to understanding the bonding and other electronic characteristics of the molecule. The energy levels themselves represent the energy values that electrons can have in the system in question. Since energy is quantized at the molecular level, only the values given by the solutions to the secular determinant are permitted. For example, in linear H_3 the secular determinant using the Huckel approximations is

$$\begin{vmatrix} \alpha - E & \beta & 0 \\ \beta & \alpha - E & \beta \\ 0 & \beta & \alpha - E \end{vmatrix} = 0$$

In this determinant, α is called a Coulomb integral, and it represents the energy of the electron when it is held by one of the atoms; it has a negative value. The parameter β is called a resonance integral, and it represents the electron being between two atoms, i.e. the wavefunctions of the two atoms overlapping. It is also negative. The rows and columns of the determinant denote the atoms

in the molecule, so entry 1-1 is atom 1 combining with itself, entry 1-2 is atoms 1 and 2 overlapping, etc. Entries 1-3 and 3-1 are zero, since atoms 1 and 3 are not adjacent and thus cannot have their wavefunctions overlap. We can use the condensation method to find this determinant:

$$A^{(1)} = \begin{pmatrix} (\alpha - E)^2 - \beta^2 & \beta^2 \\ \beta^2 & (\alpha - E)^2 - \beta^2 \end{pmatrix}$$

We then get

$$A^{(2)*} = (\alpha - E)^4 - 2\beta^2(\alpha - E)^2$$

So then

$$A^{(2)} = (\alpha - E)^3 - 2\beta^2(\alpha - E) = \det(A)$$

Setting this equal to zero we can then solve for the energy levels, obtaining

$$E = \alpha, \alpha \pm \sqrt{2}\beta$$

Thus we know what the three energy levels are for linear H_3 , and we can use this data to find the bonding energy of H_3 , which can give us information about the stability and reactivity of the molecule. For a more in-depth discussion of this topic, refer to Atkins and De Paula's *Physical Chemistry* [5].

4. PROOF OF THE CONDENSATION METHOD

In his book on the Alternating Sign Matrix Conjecture, David Bressoud outlines the general idea behind proving the condensation method, but does not go into detail as to the actual mechanics of the proof [6]. A combinatorial proof has also been published [7]. However, it is rather complicated, and so we endeavored to discover a clear proof of Charles Dodgson's method.

Proof. We shall prove the condensation method using mathematical induction.

In all cases, assume that there are no zeros in the interior of the matrix, or that they have been removed using row operations prior to beginning the process of condensation.

We will notate A_{ij}^{kl} as the 2×2 minor corresponding to the i -th and j -th rows and k -th and l -th columns of A , and a'_{ij} as the entry of the adjugate matrix in the i -th row and j -th column. In general, given A_m^n , m represents the rows and n the columns.

Base Case: 3×3 matrix.

Consider an arbitrary 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Using the condensation method, after simplification we get

$$\det(A) = \frac{\det(A_{12}^{12}) \cdot \det(A_{23}^{23}) - \det(A_{12}^{23}) \cdot \det(A_{23}^{12})}{a_{22}}$$

Applying Jacobi's Theorem, after simplifying we get

$$\det(A) = \frac{a'_{11}a'_{33} - a'_{13}a'_{31}}{a_{22}} = \frac{\det(A_{12}^{12}) \cdot \det(A_{23}^{23}) - \det(A_{12}^{23}) \cdot \det(A_{23}^{12})}{a_{22}}$$

Thus we can see that using the condensation method is equivalent to using Jacobi's Theorem for a 3×3 matrix, and thus by Jacobi's Theorem the condensation method is valid for 3×3 matrices.

Induction Hypothesis: assume that the condensation method is valid for $k \times k$ matrices ($k - 1$ steps).

It is vital to note that when using the condensation method, after 1 step we have a new matrix composed of the determinants of the 2×2 connected minors, after 2 steps we have the 3×3 determinants, and so forth; after i steps we have a matrix composed of the determinants of the $(i + 1) \times (i + 1)$ connected minors.

Now consider a $(k + 1) \times (k + 1)$ matrix. By our induction hypothesis, we can condense the matrix to a 2×2 matrix in $k - 1$ steps. We are now left with a 2×2 matrix, each entry of which is the determinant of one of the four $k \times k$ connected minors of the original matrix:

$$\begin{pmatrix} \det(A_{1 \leq i \leq k}^1) & \det(A_{1 \leq i \leq k}^2) \\ \det(A_{2 \leq i \leq k+1}^1) & \det(A_{2 \leq i \leq k+1}^2) \end{pmatrix}$$

We can now complete the last step of the condensation method: take the determinant of this 2×2 matrix, then divide the result by the interior of the previous 3×3 matrix, which will be the determinant of the interior of the original matrix. The result is thus:

$$\begin{aligned} \frac{\det(A_{1 \leq i \leq k}^1) \cdot \det(A_{2 \leq i \leq k+1}^2) - \det(A_{2 \leq i \leq k+1}^1) \cdot \det(A_{1 \leq i \leq k}^2)}{\det(A_{2 \leq i \leq k}^2)} &= \frac{a'_{11} \cdot a'_{(k+1)(k+1)} - a'_{1(k+1)} \cdot a'_{(k+1)1}}{\det(A_{*1(k+1)})} \\ &= \frac{\det(A'_{1(k+1)})}{\det(A_{*1(k+1)})} \\ &= \det(A) \text{ (Jacobi's Thm.)} \end{aligned}$$

Thus, by Jacobi's Theorem, the condensation method is valid for all $n \times n$ matrices. \square

5. DISCUSSION

The condensation method, first discovered by Charles Dodgson in 1866, provides a more efficient way of calculating determinants of large matrices. For a 5×5 matrix, it reduces the amount of computations by nearly half compared to a standard cofactor expansion method. Dodgson based his method on Jacobi's Theorem, which after some rearrangement provided a useful algorithm for finding determinants. The main weakness of the condensation method is zeros in the interior of the matrix, which create issues with the necessary divisions. However, that hurdle can often be circumvented by transforming the matrix using elementary row operations, and in cases where there are too many zeros to use the condensation method a traditional cofactor expansion is often very easy to use. The condensation method can be applied in a variety of situations, such as solving a system of linear equations with Cramer's Rule, finding the eigenvalues of a matrix, determining invertibility, and so forth. It can also be used in a quantum mechanical setting to find the energy levels of a π electron system, as was demonstrated above. The method preserves factorizations somewhat better than the cofactor method, which is important in certain situations, particularly finding the eigenvalues with the characteristic polynomial. Dodgson's method is not well known due to his general mathematical obscurity, but it can be extremely useful for finding determinants, especially of larger matrices.

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