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Lost at Sea: Introduction to Numerical Methods through Navigation

R. Corban Harwood
George Fox University, rharwood@georgefox.edu

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STUDENT VERSION
Lost at Sea: Introduction to Numerical Methods through Navigation

Richard Corban Harwood
Department of Mathematics and Applied Science
George Fox University
414 N Meridian St
Newberg OR USA

STATEMENT

The ship, El Perdido, was damaged during a storm which knocked out its main and backup power generators. Before the backup generator failed, Captain Miguel Gomez sent a distress call and the crew have been able to keep El Perdido afloat, but the ship is adrift in the Pacific Ocean off the coast of California. Thankfully, a US Coast Guard rescue operation is underway after receiving the distress call. The Coast Guard has El Perdido's last known position and has mapped out the surface water velocities in this area as slope fields for longitude (x) and latitude (y), which they have updated using historical data and estimated predictions. Since the search grid is small enough, this curved region on the surface of the earth is relatively flat.

After days of failed searches, the length of time expected for El Perdido's crew to survive has expired (calculated by the standard Cold Exposure Survivability Model). Accordingly, the Coast Guard mission coordinator downgrades the rescue operation to "Active Search Suspended, Pending Further Developments." Because the El Perdido crew had kept the ship afloat and had salvaged some of the food reserves, they are able to survive for several more days but their time is still running out.

On the fourth day adrift, El Perdido's crew is able to get a short spurt of power, in which they send another distress call and print out logs of ship velocity and acceleration from their Inertial Navigation System (INS). We join the Coast Guard rescue operation, reactivated after receiving the distress call, as they update their slope field maps and launch another search. Simultaneously,

Captain Gomez now uses dead reckoning from the INS logs to estimate his own position knowing only El Perdido's previous velocities and accelerations, in hopes that he can assist the rescuers if the crew can get the power back on.

Split your group with half helping the rescuers in Section 1 and half helping El Perdido's captain in Section 2 and then compare your results. Based upon the search radius of each rescue craft and Captain Gomez's own dead reckoning, determine if El Perdido is found on this rescue attempt.

1 Graphical Approach of the Rescuers

There are three Coast Guard rescue craft searching for El Perdido: a long-range land-based plane called the Ocean Sentry (S1), a ship-based helicopter called the Eurocopter Dolphin (D2), and a ship with a helipad referred to as a high endurance cutter (C3). All three search craft use the same surface water velocity estimates in the search grid, but each uses a different time step (in units of estimated days spent by the lost vessel as it drifted) since a greater distance is covered by a search craft with a higher speed: $\Delta t = 1$ for S1, $\Delta t = 0.4$ for D2, and $\Delta t = 0.2$ for C3. They each have El Perdido's last known position pegged at $(x(0), y(0)) = (-1, 0)$ on their search grid, but the velocity field is separated into a longitudinal slope field shown in Figure 1 with initial position $(t, x) = (0, -1)$ in Section 1.1 and a latitudinal slope field shown in Figure 2 with initial position $(t, y) = (0, 0)$ in Section 1.2. Note that the slope fields in Figures 1 and 2 are snapshots of the overall estimated velocity field and have been graphed with slightly different viewing windows.

1.1 Estimating Longitude

Traveling the fastest, the S1 plane searched a path that followed the Euler method with time step $\Delta t = 1$ on the slope fields, essentially tracking El Perdido's path using one day of estimated velocities at a time. It's estimate of the longitude has been computed for you in Table 1. Follow the steps below to compare S1's search path to the slope field, as well as plot the search paths of the D2 helicopter and C3 cutter. Along the way, answer the reflection questions (labeled Q) to check your understanding.

k	t_k	x_k	$x'_k = x_k^2 - t_k^2$
0	0	-1	1
1	1	0	-1
2	2	-1	-3
3	3	-4	7
4	4	3	-7

Table 1. Estimates of El Perdido's path by rescuer S1: longitude (x) and latitude (y) using the Euler method with $\Delta t = 1$.

1. Plot the points (t_k, x_k) from Table 1 on Figure 1 and connect neighbor points with straight line segments.

   Q: Does the slope of each line segment better match the slope field on the left or right end? Does this match your intuition that error accumulates as time progresses?

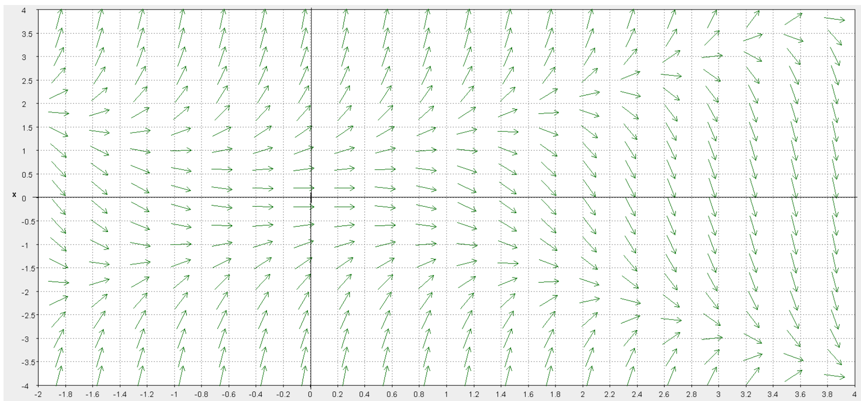

Figure 1. Display of longitudinal velocity field for rescue craft search paths.

2. Using each line as a hypotenuse of a right triangle, compute the slope (the amount of rise over run) for each and label each line with it.

   Q: What values do these slopes match?

3. On the right edge of Figure 1, circle S1's estimate of El Perdido's longitude on the fourth day as 'S1=3'.

   Q: Based upon the path of S1, do you trust this estimate?

   Now we will follow the second rescue craft, the D2 helicopter, to recreate El Perdido's path following the slope field every 0.4 days.

4. Starting again at the initial position $x(0) = -1$ on Figure 1, follow the initial slope for $\Delta t = 0.4$ and mark the right endpoint.

   Note: this will stop partway along S1's initial step.

5. Using the arrow direction nearest this point, follow the slope field for another $\Delta t = 0.4$. Then, plot the rest of D2's search path by repeating this process until you reach $t = 4$. Estimate this $x(4)$ value and label with 'D2='.

   Now we will follow the third rescue craft, the C3 high endurance cutter, to recreate El Perdido's path following the slope field every 0.2 days.

6. Plot C3's search path, again starting at the initial position $x(0) = -1$ on Figure 1 and follow the initial slope for $\Delta t = 0.2$ and mark the right endpoint. Note: this will stop partway along D2's initial step.

7. Repeat step 5 with $\Delta t = 0.2$ until you reach $t = 4$. Estimate this $x(4)$ value and label with 'C3='.

Q: Compare the search paths you plotted for S1 ($\Delta t = 1$), D2 ($\Delta t = 0.4$), and C3 ($\Delta t = 0.2$) in Figure 1. Do they appear to be converging to a solution function?

If so, highlight that curve in a different color in Figure 1. This is the exact solution. If not, check that your line segments match the arrows on the left ends and plot the paths of S1, D2, and C3 again.

1.2 Estimating Latitude

The slope field representing the latitudinal velocities can be approximated by the differential equation $y' = t - y^2,\ y(0) = 0$. Follow the steps below to compute S1's longitudinal coordinates and then follow the slope field for the D2 helicopter and C3 cutter. Along the way, answer the reflection questions (labeled Q) to check your understanding.

k	t_k	y_k	$y'_k = t_k - y_k^2$
0	0	0	0
1	1		
2	2		
3	3		
4	4		

Table 2. Latitude (y) estimates of El Perdido's path by rescuer S1 using the Euler method with $\Delta t = 1$.

1. Complete Table 2 using the Euler method approximation,

$$y_{k+1} = y_k + \Delta t\ y'_k,$$

using $\Delta t = 1$ for the S1 plane, to update each y_k latitudinal value in the next row and the differential equation to update the slope y'_k in that next row. For example,

$$y_1 = y_0 + \Delta t\ y'_0 = 0 + (1)(0) = 0, \text{ and } y'_1 = t_1 - y_1^2 = 1 - 0^2 = 1.$$

Note that the Euler method is really just the tangent line at the point (t_k, y_k), which has been rewritten from point-slope form to standard form:

$$y_{k+1} = y_k + (t_{k+1} - t_k)\ y'_k = y_k + \Delta t\ y'_k.$$

Once you have Table 2 filled in, you will repeat the steps you took in plotting and following the slope field for the longitude in Figure 1, only now using the slope field for the latitude in Figure 2.

2. Plot the latitude estimates (t_k, y_k) for S1 on the slope field in Figure 2, draw lines to connect the points, estimate the value of $y(4)$, and label it with 'S1='.

   Q: Does it look like your approximate solution fits the slope field well? Explain.

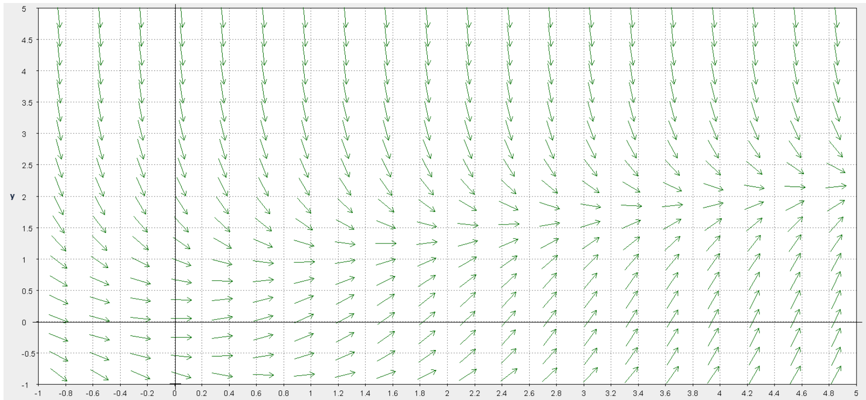

Figure 2. Display of longitudinal velocity field for rescue craft search paths.

3. Again on Figure 2, graph the search path for D2 helicopter, starting at $y(0) = 0$ and stepping with $\Delta t = 0.4$ until $t = 4$. Estimate this value of $y(4)$ and label it with 'D2='.
4. Again on Figure 2, graph the search path for C3 cutter, starting at $y(0) = 0$ and stepping with $\Delta t = 0.2$ until $t = 4$. Estimate this value of $y(4)$ and label it with 'C3='.
5. Using the slope field, draw your personal best approximation to the solution.

   Q: What things were you considering when you drew your best approximation?

Source	$x(4)$	$y(4)$	Search Radius	Distance from El Perdido	Did it See El Perdido?
S1			1		
D2			.5		
C3			.25		
El Perdido			N/A	N/A	N/A

Table 3. Position of El Perdido at $t = 4$ days estimated by rescue craft and dead reckoning.

Now, you will summarize your work in Table 3 by recording the $t = 4$ position of El Perdido estimated by the three rescue craft and comparing it with El Perdido's actual position to see if any of the rescue craft found the ship on this attempt.

6. On Table 3, record the longitude ($x(4)$) values you labeled for each rescue craft (S1, D2, C3) on Figure 1 and the final latitude ($y(4)$) values you labeled for each on Figure 2.
7. Check in with the rest of your group, sharing how you replicated the search paths of the rescue craft and asking about how they replicated the captain's hand calculations.
8. Get their best estimate of El Perdido's position $(x(4), y(4))$ at the end of Table 5. Record this point in the El Perdido row of Table 3.

9. Plot the four $(x(4), y(4))$ positions from Table 3 onto Figure 3, labeling each, to visually compare the three estimates of El Perdido's position at $t = 4$ days.

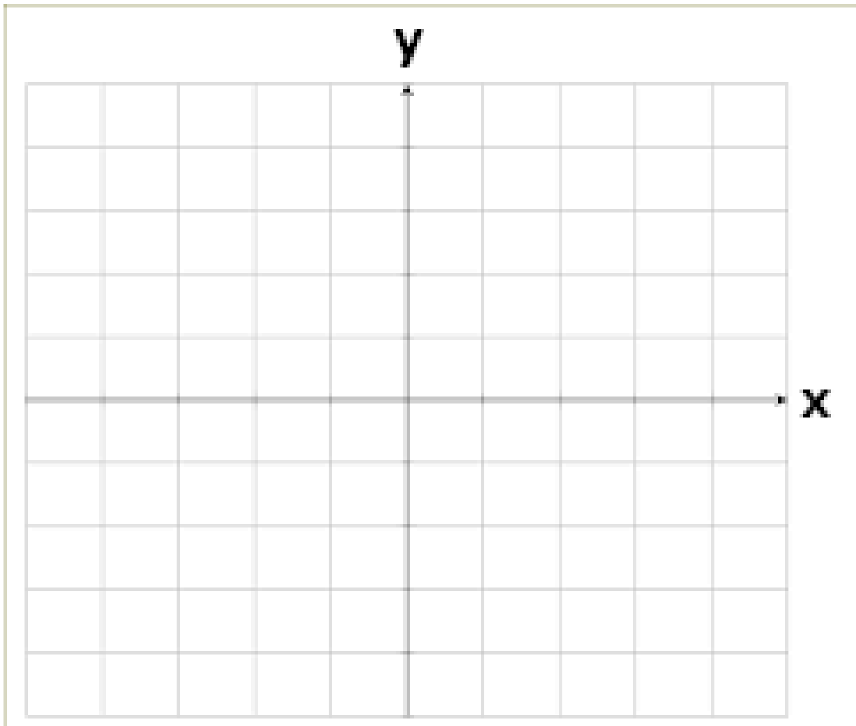

Figure 3. Longitude-Latitude grid for plotting search paths of El Perdido.

10. Calculate the distance from each search craft estimate to El Perdido's position and check if less than it's search radius to determine if this search craft found the lost ship.

2 Analytic Approach of El Perdido's Captain

Miguel Gomez, Captain of El Perdido, was able to extract velocity and acceleration information from the ship's INS (Inertial Navigation System) when his crew momentarily brought backup power online. To aid possible rescuers, he sought to estimate his own position from this data by hand. Captain Gomez selected data from the four days since the day of the storm damage, i.e. times $t = 0, 1, 2, 3, 4$, and stored them in Table 4.

2.1 Euler Dead Reckoning

Seeing that he only had one initial position, but the velocity and acceleration at every time, Captain Gomez knew he needed an equation that would allow him to compute the next position based on the current position and derivatives at either position. Starting with the definition of the derivative

of some function $x(t)$ at time t_k,

$$x'(t_k) = \lim_{\Delta t \to 0} \frac{x(t_k + \Delta t) - x(t_k)}{\Delta t},$$

he used the shorthand notation $x_k = x(t_k), x'(t_k) = x'_k, t_{k+1} = t_k + \Delta t$ to rewrite the derivative definition in the simpler form,

$$x'_k = \lim_{\Delta t \to 0} \frac{x_{k+1} - x_k}{\Delta t}.$$

Captain Gomez realized that if he started near the current position x_k and chose a small enough Δt, he could approximate the derivative without taking the limit, leaving him with only one unknown quantity.

Q: What equation did the captain get when he solved for this unknown (x_{k+1}, the next longitude value)?

$$x_{k+1} =$$

Q: What similar equation did he obtain for the next unknown latitude?

$$y_{k+1} =$$

These equations are called the *(forward) Euler method.* The captain used them to iteratively solve for x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4. Dead Reckoning is a process of estimating the position of a vessel based upon its current movement and past estimated position. When we use the Euler method to estimate such a position, we will refer to it as Euler Dead Reckoning (EDR).

Q: Compute these values (a calculator or spreadsheet is helpful), and record your values under the heading EDR in Table 4 and then do the following.

Index	Time	EDR	Data		EDR	Data		Copy Points	for Graphing
k	t_k	x_k	x'_k	x''_k	y_k	y'_k	y''_k	x_k	y_k
0	0	-1	1	-2	0	0	1	-1	0
1	1		-0.5	-1		1	0		
2	2		-1	0		1	-2		
3	3		-1.5	2		0	1		
4	4		-1	-1		1	-3		

Table 4. Velocity, acceleration, and initial position data for El Perdido extracted from the Inertial Navigation System (INS). Fill in the dead reckoning positions x_k, y_k using the Euler method (EDR).

1. Graph each (x_k, y_k) for $k = 0, 1, 2, 3, 4$ on Figure 4.
2. To represent the tangent slope at each position (x_k, y_k), graph a short arrow with the direction of the corresponding velocity vector (x'_k, y'_k) from Table 4.
3. Sketch a curve going through each position (x_k, y_k) and tangent to each slope vector (x'_k, y'_k). Label it 'EDR' for Euler Dead Reckoning.

Q: Does the EDR curve seem like a feasible path for a ship adrift? Why or why not?

Q: What else could we do to improve our estimate of the path?

2.2 Taylor Dead Reckoning

In all the excitement of finding out where he was, the captain had not noticed that the acceleration data had been ignored. Looking again at the equation for the Euler method with $x_{k+1} = x(t_k + \Delta t)$,

$$x(t_k + \Delta t) = x(t_k) + \Delta t x'(t_k),$$

Captain Gomez remembered another pattern that started out in the same way

$$x(t_k + \Delta t) = x(t_k) + \Delta t x'(t_k) + \frac{\Delta t^2}{2!} x''(t_k) + \frac{\Delta t^3}{3!} x'''(t_k) + \ldots + \frac{\Delta t^n}{n!} x^{(n)}(t_k) + \ldots,$$

called the *Taylor series.* Since he didn't have data for any higher derivatives than two, Captain Gomez truncated the Taylor series and rewrote $x(t_k) = x_k$ to obtain what is called the *(second order) Taylor method.*

Q: Write out the (second order) Taylor method for longitude (x) and latitude (y).

$$x_{k+1} =$$

$$y_{k+1} =$$

Captain Gomez now uses the Taylor method to iterate solve again for each of the longitude, x_k, and latitude, y_k values. We will refer to these approximations as Taylor Dead Reckoning (TDR). Do the following steps to replicate his work.

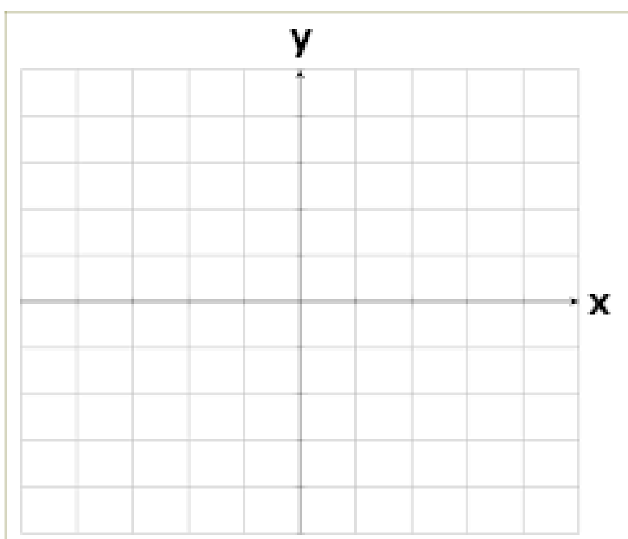

Figure 4. Longitude-Latitude grid for plotting the Euler method dead reckoning positions of El Perdido.

Index	Time	TDR	Data		TDR	Data		Copy Points	for Graphing
k	t_k	x_k	x'_k	x''_k	y_k	y'_k	y''_k	x_k	y_k
0	0	-1	1	-2	0	0	1	-1	0
1	1		-0.5	-1		1	0		
2	2		-1	0		1	-2		
3	3		-1.5	2		0	1		
4	4		-1	-1		1	-3		

Table 5. Velocity, acceleration, and initial position data for El Perdido extracted from the Inertial Navigation System (INS). Fill in the dead reckoning positions x_k, y_k using the Taylor method (TDR).

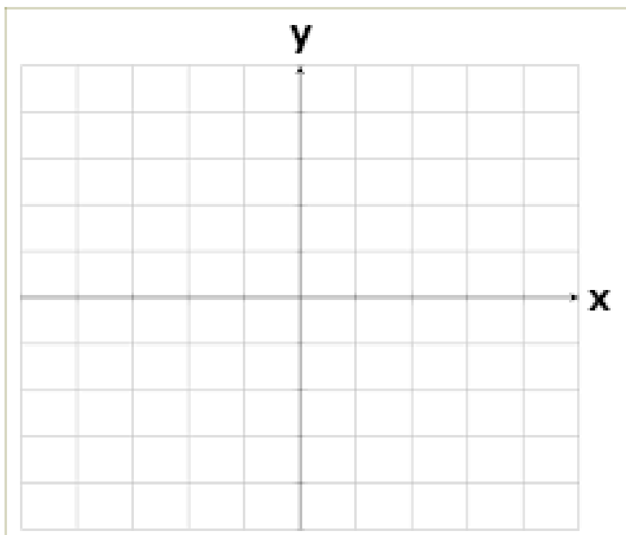

Figure 5. Longitude-Latitude grid for plotting the Taylor method dead reckoning positions of El Perdido.

1. Use the Taylor method to compute the values for x_1, x_2, x_3, x_4 and record them in Table 5 in the TDR column for x_k.
2. Similarly, compute y_1, y_2, y_3, y_4 with the Taylor method and record them in Table 5 in the TDR column for y_k.
3. Graph each (x_k, y_k) for $k = 0, 1, 2, 3, 4$ on Figure 5.
4. To represent the tangent slope at each position (x_k, y_k), graph a short arrow with the direction of the corresponding velocity vector (x'_k, y'_k) from Table 5.
5. Sketch a curve in Figure 5 going through each position (x_k, y_k) and tangent to each slope vector (x'_k, y'_k). Label it 'TDR'.

Q: Does the TDR curve seem like a feasible path for a ship adrift? Why or why not?

Q: What else could we do to improve our estimate of the path?

Now, check in with the rest of your group to find their rescuers' estimates on where El Perdido is on day four ($t = 4$).

1. Ask them how they replicated the search paths of the rescue craft and show them what you did replicating the captain's hand calculations.
2. Give them the final pair of points you computed in Table 5 for them to use in Table 3.
3. From them, obtain the positions of the 3 rescue craft S1, D2, and C3, and plot these positions on Table 5 to compare with your TDR path of El Perdido.
4. Check back with the rest of your group after they have compared the distance of each rescue craft with their respective search radius.

   Q: Did any of the rescue craft find El Perdido? Which were the closest?

3 Background

European explorers in the 15th and 16th century sailed throughout the world using a simple navigation technique called dead reckoning. They used a sand glass to precisely measure time, threw flotsam overboard at regular intervals to measure speed, and measured their direction with a magnetic compass. Today, ships, aircraft, and spacecraft continue to use dead reckoning using an Inertial Navigation System (INS) and multiple redundancies such as GPS and radar to verify their position. A modern INS tracks time digitally, measures velocity with rate gyroscopes, and measures acceleration with accelerometers.

Numerical methods provide unique perspectives to a problem (as well as quick approximations). Imagine sketching curves along all the arrows in a slope field like Figure 1 or 2. This visualizes many member functions of the family of solutions, which are each exact solutions to different initial conditions. In constrast, a numerical solution shifts between member functions of the same family at each approximation. This is demonstrated by your plot of S1's latitudinal path in Figure 2, which you computed using the Euler method in Table 1. Seeing that the slope field only matches the line segments on each left side demonstrates that the numerical solution visibly slips from one member function to another in the family of solutions with each iteration. It may jump back and forth across member functions, representing oscillations about the exact solution, and may even slip farther and farther away with each jump. These behaviors, demonstrated by S1's path in Figure 1 and Figure 2, are related to the stability of a numerical method and can be analyzed with further study.