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Nicole Enzinger
George Fox University, nenzinger@georgefox.edu

Laura Bofferding
Purdue University, lbofferding@purdue.edu

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LEVERAGING DIFFERENT PERSPECTIVES TO EXPLORE STUDENT THINKING ABOUT INTEGER ADDITION AND SUBTRACTION

Nicole M. Wessman-Enzinger
George Fox University
nmenzinger@gmail.com

Laura Bofferding
Purdue University
lbofferding@purdue.edu

This is the third meeting of a working group on student thinking about integers. The main goal of this working group includes utilizing different theoretical perspectives and methodologies in small groups to design complementary studies, where student thinking about integer addition and subtraction will be explored. This working group aims to provide a space for participants to capitalize on their differences in theoretical perspectives and methodologies to promote productive scholarly discussion about the same research topic, student thinking about integer addition and subtraction. Participants will actively engage in work that progresses towards these studies, with the intent to develop a monograph that highlights this research.

Keywords: Number Concepts and Operations; Cognition; Research Methods

A Brief History of the Working Group

The first working group on student thinking about integers convened during PME-NA 35 (Lamb et al., 2013). During this working group, facilitators shared perspectives of current research in the field on student thinking about integers. Discussion with these speakers and participants revolved around the presentations and what "Integer Sense" entails. The work initiated at PME-NA 35 continued at the joint PME 38 and PME-NA 36 meetings (Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled, 2014). At this meeting, the organizers presented an extensive review of all of the integer articles from the PME and PME-NA proceedings. Further, they shared and discussed perspectives on integer research stemming from seminal work on integers. The group concluded with a discussion on next directions for collaborative research on the teaching and learning of integers. Responding to the need for collaborative research, this working group proposes a collaborative research project that welcomes all perspectives on student thinking about integer addition and subtraction and provides a platform that embraces these collective differences.

Relevance to Psychology of Mathematics Education

Compared to research on whole number addition and subtraction, research on student thinking about integer addition and subtraction is fairly limited. Perhaps for this reason, there is increased interest in student thinking about integer addition and subtraction in our field (e.g., Bishop et al., 2014a, 2014b; Author, 2014; Stephan & Akuyz, 2012). However, research on student thinking about integer addition and subtraction has been conducted, and even represented at PME and PME-NA, for over three decades (e.g., Bell, 1982; Gallardo, 2003; Marthe, 1979; Peled, Mukhodpadhyay, & Resnick, 1989). This points to a need for research on student thinking about integer addition and subtraction to begin building bridges to connect the research. One way to do this will be to discuss similarities and differences when using different theoretical lenses or analyses on similar topics (e.g., Lewis, 2008).

For guidance as a field interested in the growth of research on student thinking about integers, it is helpful to turn to well-established agendas, like the research on student thinking about whole number operations, and reflect on how these agendas have flourished and have became well-connected. Research on student thinking about whole number operations grew and became the most proliferated and well-connected area of research (Mathematics Learning Study Committee, 2001) with researchers taking different perspectives on similar topics. The 1970s involved investigations into how young children counted on or solved different types of word problems (e.g., Jermam, 1970; Steffe & Johnson, 1971). Research on student thinking was often focused on "basic skills," accuracy, and speed (Jermam, 1970; Bright, Harvey, & Wheeler, 1979).

By the 1980s and 1990s, research exploded on student thinking about whole number operations, where most of the scholarly discussion revolved around student-invented strategies and different problem types (e.g., Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, & Moser, 1984; Fuson et al., 1997). As this influx in research on student thinking about whole numbers increased, researchers responded to the multiple perspectives on student thinking. For example, Cobb (1985) reacted to three different papers from Baroody (1984), Carpenter & Moser (1984), and Fuson (1984) that appeared in *The Journal for Research in Mathematics Education*. We can now look across these agendas and compare and contrast the findings and perspectives, and even categorize these perspectives by agenda (e.g., Cognitively Guided Instruction (CGI), Conceptually Based Instruction Project (CBI), Problem Centered Mathematics Project (PCMP), Stages of Early Arithmetical Learning (SEAL), Supporting Ten Structures (STST)).

By the 2000s, the similarities and differences of these different theoretical perspectives and research methodologies were embraced and projected the field forward, not only in the area of whole number arithmetic but in many other areas, such as early algebra (e.g., Carpenter, Franke, Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006), understanding of the equal sign (e.g., Jone & Pratt, 2011), rational numbers (e.g., Empson & Levi, 2011; Steffe & Olive, 2010), and even preservice teacher education (e.g., Vacc & Bright, 1999).

If we compare the development of research and research agendas on student thinking about integers to the field of whole numbers, we can learn that embracing multiple perspectives is productive and insightful. Similar to the increased interest in student thinking about whole number arithmetic of the 80s and 90s, we are currently positioned to respond to this increased research interest on student thinking about integers. Drawing upon these productive comparisons and research, this working group aims to establish a space for those interested in researching student thinking about integer addition and subtraction. Based on past participation in the working group, we anticipate that participants will represent a variety of theoretical perspectives, which will fuel a set of complementary studies and continued discussion about similarities and differences in our investigations.

Theoretical Perspectives & Methodological Approaches

Across the PME and PME-NA proceedings and recent journal articles, researchers have presented work around negative integers from a variety of perspectives and using different methodological approaches. Some or all of these may play a role in the working group discussion, studies, and final products. We present a few examples:

Integer sense. Both Gallardo (2002) and Bishop et al. (2014a) point to ways that that we can think about and use negative integers (see Table 1). Gallardo based her framework for interpreting negative integers on historical analyses of the topic and clinical interviews with middle-schoolers. Bishop et al. based their interpretations from the literature and mathematical reflections.

Rather than focusing on the different ways of interpreting integers, Kilhamn (2009) theorized about what number sense is in relation to concepts involving integers. These components include "intuitions about numbers and arithmetic" (p. 331), the "ability to make numerical magnitude comparisons" (p. 332), the "ability to recognize benchmark numbers and number patterns" (p. 333), and "possessing knowledge of the effects of operations on numbers" (p. 334).

Table 1: Comparisons and Interpretations of Gallardo (2002) and Bishop et al. (2014a)

Gallardo (2002) interpretations of negative numbers (p. 179)	Bishop et al. (2014a) interpretations of -5 (p. 20)	Reflections
Subtrahend "where the notion of number is subordinate to the magnitude (for example, in a – b, a is always greater than b where a and b are natural numbers"	"An action of removing 5 from a set"	Removing five from a set matches closely to interpreting a negative number as subtracting a positive number.
Relative or Directed Number "where the idea or opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domains"	"The location on a number line (coordinate plane, etc.) 5 units to the left of, or below, 0" "An action of moving 5 units left or five units down" "A debt of $5 is also a directed number; it is the opposite of a credit of $5."	Placing a negative number on a number line allows one to interpret the negative number as a relative number or a directed number. Debt can be interpreted as direction. Or, -5 can be a relative number that represents a loss of five dollars.
Isolated Number the result "of an operation or as the solution to a problem or equation"	"The integer between -6 and -4"	The negative number may be treated as a symbolic number that has order.
Formal Negative Number "a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers)"	"Describing the equivalence class [(0,5)] in which we define (a, b) to mean a – b, and all other ordered pairs (a,b) such that a + 5 = 0 include (1, 6), (2, 7), (100, 105), and all other ordered pairs (a,b) such at a + 5 = 0 + b for a,b that $\in \mathbb{N}$. [More formally, we can write (0, 5) ~ (a, b).]"	The negative number can be thought of in more formalized ways. For example, -5 is compared to an equivalence class. We can also talk about the additive group of the integers or the ring of the integer and how integers are not a field because the multiplicative inverses are not all integers.

Integer number line and conceptual change. Using an experimental design involving pre- and post-test task-based interviews as well as instruction around different aspects of integer concepts, Bofferding (2014) identified different conceptions children have of integers. Aligned with a conceptual change paradigm (Vosniadou, 1994), the conceptions fall along a continuum where initial ideas about negative numbers arise from students' conceptions of whole numbers and progress to more formal understanding. Although the categories and concepts explored in the research support Kilhamn's (2009) ideas of number sense, Bofferding's interpretation of students' work focuses on how their conceptions arise and become differentiated from their whole number understanding and when planning for instruction focuses on how to effectively bridge learning of whole number and integer concepts. Further exploration of students' developing integer conceptions would benefit from the use of additional research methods, such as a teaching experiment (Steffe & Thompson, 2000) or microgenetic analysis (Siegler, 1996), which could further clarify what portions of the instruction or work with integers influenced students to change their thinking in Bofferding's study.

Conceptual models. Wessman-Enzinger & Mooney (2014) found that when children posed stories for integer addition and subtraction problem types the students' reasoning could be classified into the Conceptual Models for Integer Addition and Subtraction (CMIAS). The five CMIAS described are Bookkeeping, Counterbalance, Translation, Relativity, and Rule. With *Bookkeeping,* the integers are treated as gains and losses, and zero represents neither a gain nor a loss. For example, students posed stories that involved gains and losses of candy bars. And, other students used "needs" and "wants" of various other items, like baseball cards, to represent the negative integers. With *Counterbalance,* the integers are treated as neutralizing each other, and zero represents a status of neutralization. A distinguishing feature of Counterbalance from Bookkeeping is that the quantities always remain present with Counterbalance. For example, consider -2 + 3 = 1, which can be represented by two electrons and three protons, where there is an electrical charge of 1. Although there is an electrical charge of 1, the two electrons and three protons still remain present. With *Translation,* the integers are treated as a vector or with movement. Zero in Translation represents either the position or no movement. With *Relativity* the integers are treated as a comparison to an unknown referent. Zero represents the unknown referent. For example, for -5 + -10 = -15 a student posed the story, "Say you are down five runs in the first inning of a baseball game. And you end up losing by fifteen runs. You would have to have ten runs in the other innings to be down by fifteen runs" (Wessman-Enzinger & Mooney, 2014, p. 203). In this story, the actual score of the game is unknown. The integers are used as relative numbers to the unknown referent, the score of the tied game. Relativity is related to Translation, but both movement and the dual-role of the zero in the model distinguish Translation. Although related to Gallardo's (2002) interpretation of directed number and relative number, the CMIAS distinguishes the use of the integers here. With *Rule* the integers are treated with a procedural rule about signs. Wessman-Enzinger and Mooneys' interpretations of student thinking about integers is that hidden behind the use of contexts are implicit mathematical meanings. And, the utilization of these contexts is isomorphic to mathematical uses of integers. Further investigation into ways that students respond differently to contextualized problems, promoting different CMIAS, could be supported by both task-based interviews (Goldin, 2000) or teaching experiments (Steffe & Thompson, 2000).

Goals of Working Group

During this working group, one of the goals will be to establish small-groups that will work collaboratively together on a research project of their creation. Each of these groups will begin by analyzing similar video data provided by the organizers. This will initiate discussion on ways that using different theoretical perspectives and analysis highlight similarities and differences in student thinking about integer addition and subtraction. However, the main aim of this working group will be to begin formulating research questions and begin designing small studies that are each related to each other. It is the hope of this working group that each of these small studies would be conducted in 2016 and that participants will begin analysis in 2016 as well. Participants could share their initial results at a future meeting so that the group can begin to make comparisons and larger conclusions about student thinking. Additionally, the working group will select a journal to propose a monograph to that illustrates and embraces these different perspectives on making sense of student thinking about integer addition and subtraction.

Extending the Previous Work from Past Working Groups

Both of the previous working groups, PME-NA 35 and PME 38/PME-NA 36, were discussion-oriented. Drawing on colleagues' work and literature reviews, the focal point of each of these discussions was on student thinking about integers and integer addition and subtraction. Each of these working groups concluded with participants eager to work collaboratively on a project. This working group extends the work from these previous groups by presenting a collaborative project that supports connecting our multiple research agendas, inviting all, whether neophytes or experienced researchers, interested in student thinking about integer addition and subtraction to participate.

Plan of Working Group

Plan for Session 1

The first session will begin with a brief overview on the history of the working group for any new members attending. The facilitators will also give a brief update from the joint PME 38 and PME-NA 36 meetings. Then, the session will transition to introductions among participants. The participants will briefly share their interests in integer addition and subtraction research and the general theoretical perspectives and methods they employ. The facilitators will ask participants to break into small groups. The facilitators will show a short video clip of a student solving an integer open number sentence. Discussion will begin with the following question:

1. Using your preferred theoretical perspective on student thinking about integer addition and subtraction (e.g., Metaphorical Reasoning, Mental Models (MM), Ways of Reasoning (WoR), Conceptual Models for Integer Addition and Subtraction (CMIAS)), how would you make sense of or describe the student thinking in this video?

This question will be discussed extensively in small-groups first, and then we will transition to whole-group discussion together. In whole-groups we will discuss the following question:

2. What similarities and differences are present in our discussion about student thinking?

Plan for Session 2

The second session will begin with a brief summary of the previous day's discussion. The facilitators will then describe the proposed collective study where participants will work in small groups to design and implement small studies in the domain of integer addition and subtraction. Together, the group will brainstorm a central, overarching research question to drive the individual studies. The participants will then spend time in their small-groups deciding a topic and research question for their mini-study. At the end of the session the participants will submit their status and progress in a Google survey document so that the facilitators are aware of progress or issues within the group.

Plan for Session 3

The third session will begin with the facilitators engaging the participants in discussion about what was reported in the Google survey document. Discussion will also revolve around expectations for the studies and ways to keep active as a group throughout the year (e.g., scheduling virtual check-ins, meeting at other conferences throughout the year). Then, the remainder of the time will be spent in small groups actively working on the planning and the logistics of the implementation of their small studies.

Anticipated Follow-up Activities

The facilitators will promote active engagement throughout the year in at least two ways. First, the facilitators will provide a shared Google spreadsheet listing our groups' goals to participants. The shared Google spreadsheet will have the small-groups listed as the rows and the collectively-decided group goals (e.g., topic, research questions, theoretical lens, participants, data collection, data analysis) as the columns, which will be modified during the third session. The group will use this shared spreadsheet as a place to report dates when they complete a goal and as a way to see the progress of the other groups in comparison. Second, the small groups will be encouraged to plan at least one Skype session with the facilitators. That way, the facilitators will be aware of each of the group's progress. This will be important for coordinating both the next PME-NA working group session, as well as planning the monograph, where each of these studies become chapters.

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