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Abstract

In this article, we explore how students attempt to bridge from their whole number reasoning to integer reasoning as they solve subtraction problems involving negative numbers. Based on interviews with students ranging from first graders to preservice teachers, we identify two overarching strategies: making connections to known problem types and leveraging conceptions of subtraction. Their initial connections suggest that rather than identifying the best instructional models to teach integer concepts, we should focus on identifying integer instructional models that build on the potentially productive connections that students' already make; we propose an example of one such form of instruction.

Keywords: negative numbers; pre-service teachers; numerical reasoning; subtraction problems; elementary mathematics education

Introduction

Although integer research spans both student thinking and instruction, historically, there has been a primary focus on integer instructional models and

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representations (e.g., Arcavi, Bruckheimer, & Ben-Zvi, 1982; Dickinson & Eade, 2004)–with a secondary focus on how students reason with them (e.g., Janvier, 1985; Liebeck, 1990; Linchevski & Williams, 1999). Furthermore, there has been a subtle obsession about finding a model that is best or can seamlessly handle all integer arithmetic (e.g., Janvier, 1985; Liebeck, 1990; Schwarz, Kohn, & Resnick, 1993). Some studies have focused on presenting or comparing methods for teaching integer addition and subtraction (e.g., Hitchcock, 1997; Rodd, 1998). These methods include cancellation using two colors of chips (Liebeck, 1990), trains (Schwarz et al., 1993), balloons and weights (Janvier, 1985), or double abaci (Linchevski & Williams, 1999) and movement using number lines (Liebeck, 1990; Nicodemus, 1993; Herbst, 1997). Other variants include a focus on net worth (Stephan & Akyuz, 2012) or symmetry (Tsang, Blair, Bofferding & Schwartz, 2015). In some cases, researchers argue that no one ideal model exists (e.g., Vig, Murray, Star, 2014). Indeed, results of the instructional studies are mixed, sometimes favoring one method or another, but most identify areas where students have difficulty.

Researchers have also investigated student thinking about negative integers (Bofferding, 2014; Peled et al., 1989) and operations involving negatives from kindergarten to preservice education (Bishop et al., 2014a; Bofferding, 2010, 2014; Bofferding & Richardson, 2013; Peled et al., 1989; Wessman-Enzinger & Mooney, 2014). One area that causes persistent cognitive conflict is subtraction problems involving negatives (Wheeler, Nesher, Bell, & Gattegno, 1981). Students who can correctly solve other integer arithmetic problems often struggle to make sense of problems such as 3-5 (Murray, 1985). Further, magnitude reasoning breaks down when transitioning from whole number to integer arithmetic (Hefendehl-Hebeker, 1991); that is, subtraction does not always result in a smaller number.

One reason why there may be such variability in the effectiveness of integer instructional models is that students build off of their whole number knowledge in various ways as they work to make sense of negative integers (Bofferding, 2014). For example, they might interpret negative integers as only qualitatively different than positive integers and use them as if they had equivalent values (Bofferding, 2014; Peled et al., 1989). Other students might interpret the negative sign as a subtraction sign and interpret numbers as amounts taken away from themselves, equivalent to zero (Bofferding, 2014; Hughes, 1986; Lamb et al., 2012; Murray, 1985). Additionally, some students might order negative integers based on the notion that numbers that are more negative are greater than numbers that are less negative; therefore, they assert that $-4 > -2$ (Bofferding, 2014). Finally, some students can reason that negatives closer to zero are larger (Bofferding, 2014) or closer to being out of debt in a money context (Stephan & Akyuz, 2012). Often negative integers are not introduced until later grades in school mathematics, despite young children's capability of reasoning with negatives integers (e.g., Bishop et al. 2014a, 2014b; Bofferding, 2014; Davidson, 1987). Yet, negative integers provide a productive space for robust mathematical discussion (Featherstone, 2000). When negative integers are introduced later in school mathematics, students' ideas that adding makes larger and subtracting makes smaller (and other whole number reasoning) may be reinforced for years. Although ideas and conjectures made from using whole number reasoning do not always hold for integer reasoning, drawing upon whole number reasoning in productive ways could help students expand and develop their emerging integer reasoning.

Our goal here is to present examples of how students connect their whole number reasoning to integer problems so that we can begin to explore the range of instruction that could support productive connections (rather than starting from a particular instructional model).

Our Data

Between the two of us, we interviewed and posed negative integer arithmetic problems to first graders, second graders, fifth graders, and pre-service teachers (PSTs) as part of separate research projects. After discussing our data and results from these projects, we combined our data, which allowed us to look at how students made connections between whole number and integer subtraction across the ages. Our theoretical lens for looking across age groups and combining data is based on the notion that experiences, more so than development, influence reasoning about integers (Bofferding, 2014; Bruner, 1960; Fischbein, 1987). Furthermore, exploring reasoning across a range of familiarity levels with integers made it likely that we would capture the variety of ways students connect integer reasoning to whole number reasoning in potentially productive ways.

In particular, we focus the discussion here on subtraction problems involving negative integers, as they are considered the most difficult (Wheeler et al., 1981). Collectively, students solved problems posed without a context (e.g., -5 – -3= □; 1– 4 = □; 4 – □= 8) and within a variety of story contexts (e.g., points in a game). We used the combined data from our studies to explicitly look at the ways that our students

(1st graders - PSTs) connected their whole number reasoning to integer reasoning. Frequently, students explained this connection explicitly; however, we also looked for instances where they made a connection implicitly, such as when using a method they would normally use for whole number operations (e.g., changing a subtraction problem into a missing addend problem).

Connecting Whole Number Reasoning to Integer Reasoning

There are several ways that students draw on their whole number reasoning when working with negative integers, and they fall into two broad categories: making connections among problem types and making connections to conceptions of subtraction.

Connections Based on Problem Types

When making connections based on problem types, students often referred to or made analogies to similar positive number problems. Bell pointed out that, "Adding two negatives is seen as adding two quantities of the same kind" (Wheeler et al., 1981, p. 28) and asserted that subtracting could be the same. This reasoning parallels whole number reasoning and is illustrated below, along with other connections students made.

Problem type: a – a = 0, where a < 0 (e.g., -6 – -6). Problem types such as -4 – -4 = 0 are a useful problem type for helping students connect their whole number reasoning to integer reasoning. For example, when solving -8 – -8, several first graders explained that if you have negative eight and take it all away, then you are left with zero, similar to their reasoning for why 8 – 8 = 0. One PST generalized this further, stating, "I know that when you subtract a number from itself you get zero." Students are even able to use this reasoning to solve the more complicated unknown minuend or unknown subtrahend problems. For ☐– -5 = 0, a fifth grader reasoned, "Negative five minus negative five, that would be zero...Because if you have five and you got rid of five, that would mean that you had zero left." Although students have trouble describing what a negative number of things means, those who correctly solve these types of problems generally connect to their prior knowledge of whole numbers in a way that uses quantities more abstractly.

Problem type: x – y, where x, y < 0 and |x| > |y| (e.g., -9 – -6). When subtracting two negatives, one of the most natural "spontaneous generalisations" (Wheeler, Pearla, Bell, & Gattengo, 1981, p. 28) students make is that their answer will be negative - similar to how they previously got a positive answer when subtracting two positives. For example, on -4 – -3 = ☐, a second grader reasoned, "I put four, four fingers like negative (holds up four fingers), and then I took away, three away (puts down three fingers), and then I had negative one." This student used fingers as if working with whole numbers, but she remembered that the quantities involved were negative.

This form of reasoning by making analogies to positive number problems also works well when drawing on contexts as in the following problem:

> Brianna started with playing cards worth -4 points. Her opponent took a -3 point card from her. What is Brianna's new score?

When explaining her strategy, one elementary PST said, "Cause if I think about having four things and someone takes away three, then I'm gonna have one left. If it's negatives, it works the same way." A similar focus on quantities can help students make sense of missing minuend problems, as shown through one PST's explanation for why the box in ☐– -3 = -1 should be -4:

> It has to be more negatives, because the answer is a negative one. And I know that I just have to have, like one more than three because I'm taking away three...I need one left, so I have to have one more than three...it has to be a negative four.

She frequently used positive number language to talk about the relations among the numbers and then mapped her answer back onto the negatives.

Problem type: x – y, where x, y > 0 or x, y < 0 and |x| < |y| (e.g., 6 – 9 or -6 – -9). Unlike the previous two problem types, where both integers are negative and the minuend is larger in absolute value than the subtrahend, students often have difficulty productively using reasoning about quantities when the subtrahend is larger in absolute value than the minuend. On problems such as 6 – 9, many K-5 students in our sample indicated, "You cannot do this" or answered, "0." However, students who could solve problems such as -2 – -2 = 0 using whole number reasoning could leverage their knowledge to solve more difficult ones. For example, one first grader solved -4 – -7 by breaking apart the -7 into -4 and -3: "Because negative four minus four would be zero, but since four plus three equals seven...it would probably be three more than zero." The student related breaking the problem apart to a similar whole number problem and reasoning about the parts. Another fifth grader did this while explaining how the missing number in ☐– -2 = 1 should be negative one. He first pointed out, "Negative one is before two." He then compared ☐– -2 = 1 to 1 – 2 = -1: "Like if you had one minus two that would equal negative one. But, now it's a negative one minus a negative two equals regular one. It's like ... flipping it around kind of."

Problem type: x – y, where x > 0 and y < 0 (e.g., 6 – -9). Of all of the problems involving subtracting a negative, subtracting a negative from a positive is often the least intuitive for students when trying to connect it to other problem types involving whole number reasoning because of the strong emphasis on the take-away meaning of subtraction in the elementary grades. Bell asserted that problem types like these, “cannot be dealt with correctly in such a system to the ‘like quantity’ notion” (Wheeler et al., 1981, pp. 28–29). Rather than drawing upon connections to other problem types, our students mostly solved this particular problem type by making connections to whole number reasoning based on their conceptions of subtraction.

Connections Based on Conceptions of Subtraction

Bell advocated that the teaching and learning of operations with negatives should be built upon conceptualizations of subtraction (Wheeler et al., 1981). The following descriptions provide insight into student thinking built upon different conceptualizations of subtraction.

Subtraction as take away or comparison. Although perhaps the least generalizable conceptualization of subtraction to use for integer operations, some students relied on the “take away” or comparison meaning of subtraction and productively used quantity-related representations to solve the problems. For example, when solving 3 – 9=-6, one first grader started by raising three fingers and putting them down sequentially while counting to nine: “One, two, three.” At this point all of his fingers were down to show zero, so he continued by putting up fingers sequentially while continuing to count, “Four, five, six, seven, eight, nine. (Looks at six fingers up.) Negative six.” Another fifth grader solved 12 – 18 using tallies (see Figure 1) and comparing two positive quantities:

Well, I did twelve (points at the green tallies.) as the twelve (points at 12 in the number sentence). And then, I did eighteen (points at the pink tallies). And this is twelve (points at the left side of the pink tallies). So I (takes hand and covers green and pink tallies) knew that the six extra ones (still covering both sets of twelve tallies, uses right hand to point at the uncovered pink tallies) were the answer.

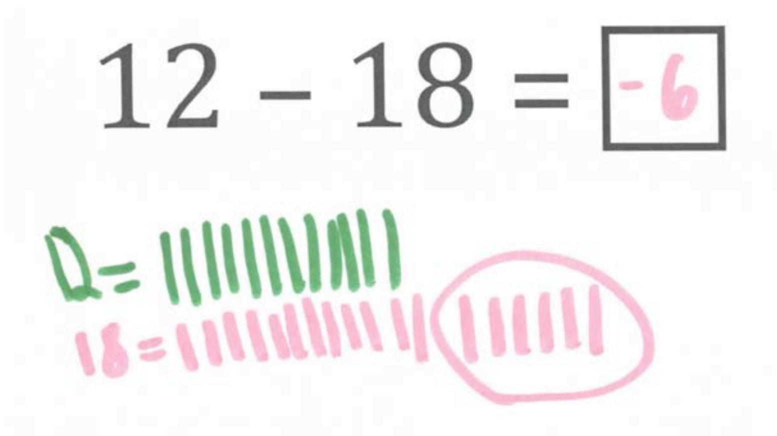

Figure 1. A fifth grader's drawing for solving 12 – 18 = ☐

In this example, the fifth grader was able to compare the two positive quantities and recognize that the "six extra ones were the answer," resulting in a negative solution. She drew upon her whole number reasoning as she modeled both quantities discretely and compared them, but extended this reasoning to the integers.

Subtraction as missing addend. When students encounter a problem they have not seen before they can have "spontaneous" and creative solutions (Wheeler et al., 1981). In an initial session on solving integer operations, a fifth-grader solved ☐– -2 = 1. She re-structured the problem, solving 1 + -2 = ☐instead of ☐– -2 = 1:

Fifth-grader: (Writes 1 + -2 = -1 vertically). I did one...I did it backwards.

Researcher: Ok. Can you explain that?

Fifth-grader: Negative one. I did one plus two, negative two I counted it up. I counted one plus negative two up, and I got negative one.

Researcher: What do you mean that you counted from one to negative two up? Can you explain that?

Fifth-grader: Well, I had negative two is below zero. So, I did negative two and I added one and I got one.

This strategy built off of her previous whole number reasoning where subtraction can be solved as finding a missing addend, and she used this strategy productively to solve a notoriously challenging integer subtraction problem the first time she encountered it. Likewise, one second grader started with a missing addend problem and used it to find the answer to two corresponding subtraction problems by using fact families. After solving ☐+ -3 = 6, she then wrote that 6 – -3 = 9 and 6 – 9 = -3.

Subtraction as directed distance. When subtracting, students can interpret the situation as a directed distance between two numbers (Tillema, 2012), which act as endpoints. In this case, the subtraction sign indicates a comparison between two numbers, and the negative sign indicates a number that is less than zero or to the left of a reference point. Comparison situations are a natural way to encourage directed distance reasoning as seen by one PST's solution to the word problem representing 6 – -7: Andy has six points. Joan has negative seven points. How many more points does the winner have than the loser? After determining that Andy was winning, the PST reasoned, "To get to zero, Joan has to like gain seven more points, and then to get to where Andy is…she has to gain six more. So it's seven plus six, thirteen." As seen in this strategy, calculating the distances to and from zero, a prominent reference point, is a common strategy with this reasoning. If the question had been reworded to ask, "How many points does Andy need to tie with Joan?" then the distance would have been 13 in the negative direction, or -13 points.

Subtraction as directed movement. Bell indicated that students will often try to use directed movement to solve problems such as -3 – -10 by starting at a point and counting on or back, but they will have difficulty interpreting the meaning of two minus signs (Wheeler et al., 1981, p. 28). Students typically interpret subtraction as "getting less" of something, which they equate with counting down (or getting less in the positives). This meaning can be expanded with directions and movements on the number line. Movements are less positive if they move away from positive infinity (or toward negative infinity); likewise, movements are less negative if they move away from negative infinity (or toward positive infinity). To subtract a negative number, a student needs to move in a direction that is less (indicated by the subtraction sign) negative (indicated by the negative sign). Even first graders used reasoning related to directed magnitudes to figure out the answers to problems such as -4 – -7. One first grader started at -4, then counted 7 through zero to get an answer of 3, explaining, "I got less negative, but it went to positive." Similar to when subtracting a smaller negative from a larger one, this student knew the counting would be in a less negative direction (getting smaller in the negatives). Using the order of integers, the student was able to move past zero and continue counting to get a positive answer. A second grader further explained, "Since it's a take away and they're both negative, we have to go farther from the negatives and closer to the positives." A secondary pre-service teacher used similar reasoning for 9 – -2: "I have nine, but I want to take away a minus two, so it'd go in the positive direction. It'd be eleven."

Discussion

Because they spend the majority of their school lives focusing on operations with whole numbers, students naturally build off of their whole number reasoning as they begin to incorporate negative integers into their conceptions (Bofferding, 2014). Our data from students across all stages of learning highlight several examples of potentially productive connections students make when subtracting negative integers, through making connections among problem types and leveraging conceptions about subtraction. Here we discuss these strategies more broadly, explore implications, and provide suggestions for exploring these connections in instruction and future research.

Leveraging Problem Types

Across the grades, there were striking similarities in students' reasoning for solving the problems, especially in terms of the analogies they made among problem types. Analogies were most productive when made by students between subtracting two positives, where the subtrahend is equal to or smaller than the minuend (e.g., 6 – 3 or 6 – 6), and subtracting two negatives, where the subtrahend is equal to or less negative than the minuend (e.g., -6 – -3 or -6 – -6). Indeed, these problems require less knowledge about negatives and are sometimes considered the easiest (Human & Murray, 1987); therefore, they could provide a gentle transition to integer operations from whole number operations. Students' eagerness to make analogies among problem types suggests that a focus on exploring contrasting cases and worked examples could be particularly helpful (e.g., Rittle-Johnson & Star, 2011). By exploring how problem types are similar or different, students could develop a deeper understanding of when their analogies do or do not hold and have the opportunity to develop other productive analogies.

When describing the connections among problem types, the older students often had more detailed justifications -- with some exceptions. Having younger students describe their justifications could push them to consider the relations among problems more deeply; however, future research should explore more clearly the role justification plays in students' analogies. Future research should also explore how analogies evolve during and after students spend time solving problems with negative integers.

Leveraging the Multiple Meanings of Subtraction

Our data suggest that emphasizing the multiple meanings of subtraction, while important for whole number reasoning, can support later integer understanding. In particular, students productively used the distance and directed movement meanings of subtraction to solve complex problems, such as 6 – -9. Generally, the older students were more likely to use this method. Two possible reasons for this trend may be because the distance meaning of subtraction is not emphasized much in the early grades (e.g., Selter & Prediger, 2012) or because younger students have more difficulty coordinating both direction and distance simultaneously. Future research could explore whether a focus on the distance meaning of subtraction with negative integers could be productive in earlier grades as well.

Connections to Integer Models

Across the grades, many students used objects to solve some of the problem types (e.g., -7 – -5). Models that involve tangible items, such as the chip model (e.g., Kajander, Mason, Taylor, Doolittle, Boland, Jarvis, & Maciejewski, 2010), could be helpful in these instances to build on students' natural tendencies and illustrate their strategies. Likewise, using number lines can capitalize on students' tendencies to utilize directed movement and distances to reason about integer subtraction problems. In this way, the models arise from students' thinking and help illustrate their thinking rather than being imposed on them. Once students understand the models as used to represent their thinking, they can then be extended to other problem types.

A Caution on Avoiding Students' Integer Intuitions

Whether or not students' negative integer connections to whole numbers are deemed useful, they are inevitable. Avoiding intuitions leaves students free to apply their thinking to other situations where the connections are *not productive*, and if they are not addressed, the overgeneralized connections can be hard to overcome later. A classic example is students' assertion that a problem such as 3 – 5 is equivalent to 5 – 3 (Bofferding, 2011), even if they can solve other negative integer problems (Murray, 1985). The complexity of the teaching and learning of integers is that building off of whole number reasoning alone is not enough. In our data, there were some instances where students' intuitions were incorrect and would need to be addressed.

A common overgeneralization students made was that problems with a negative sign would have negative answers (Wheeler, Nesher, Bell, & Gattegno, 1981). For example, when solving 8 – - 7 = -1, a fifth grader explained, "I did eight minus seven, then I added the negative." Another fifth grader also overgeneralized the commutative property of integer addition when solving -5 – 9 = -4:

> I switched them around...nine take away five...is four, but since this (referring to negative five) was a negative at first I just knew it had to be here (points to answer).

Another overgeneralization they made was that you cannot subtract a larger negative number from a smaller negative number (just as many thought they could not subtract a larger positive number from a smaller positive number). When solving -4 – -7 = 0, a first grader stated, "Negative seven (points to -7) is more than negative four (points to -4), so you can't minus it." In many cases, students' overgeneralizations were tied to their reliance on the take-away meaning of subtraction. Helping students confront these conceptions is important for their developing integer understanding, and we need to continue investigating effective instruction to address this issue; helping them think about the other meanings of subtraction (in addition to the take-away meaning) could be helpful.

Links to Instruction

This paper serves as a resource on ways that students can think about subtracting negative numbers, and we can leverage the different problem types and conceptions of subtraction to support students' understanding of negative numbers in targeted ways.

Focusing on directed magnitudes, especially together with finding the directed distance, can help students extend their understanding of integer addition and subtraction and help them develop consistent meanings for the subtraction and negative signs. Having students look for analogous situations among negative number and positive number problem types (e.g., -5 – -3 = -2 and 5 – 3 = 2) could further assist their efforts to make generalizations and connections among positive and negative operations.

A blend of problems focusing on the number sequence and quantities (e.g., finding the distance, comparing discrete quantities) provides a powerful combination for helping students make sense of why subtracting a negative number is equivalent to adding a positive number.

Using both problem type and conceptions of subtraction is a powerful pairing for instruction on negative integers. One way to integrate the problem type and conceptions of subtraction while building on whole number reasoning is to use number strings, a series of related problems (DiBrienza & Shevell, 1998; Kazemi, Franke, & Lampert, 2009). By asking for students' reasoning and using number line representations in targeted ways, teachers can help students make connections among the problems. For each problem, the teacher writes it on the board, elicits students' answers and strategies, illustrates their thinking, and asks them to think about patterns among the problems. In conclusion, we propose a potential number string (see Table 1) that begins with whole numbers only and transitions to using whole numbers and integers to connect students' discrete quantitative thinking to directed movement and distance and to "link directly with the pupil's thought process" upon encountering a new problem type (Wheeler et al., 1981, p. 29).

Table 1

Example number string for subtraction with negative numbers

Problems	Directed Movement	Directed Distance
11 – 8 = ☐	If students talk about taking away 8: Show 11 on an empty number line, have students focus on	If students talk about counting up from 8: Show 8 and 11 on the empty number line and show that the

	which direction they move when taking away 8, and show that jumping back 8 ends at 3.	distance from 8 to 11 is 3 units in the positive direction.
6 – 1 = ☐	If not addressed by students' methods, have students articulate which direction they move on the number line and show the jump.	If not addressed by students' methods, have students help illustrate the distance from the subtrahend to the minuend.
6 – 0 = ☐	See above.	See above.
6 – -1 = ☐ 6 – -5 = ☐ 11 – -7 = ☐	Make connections to the prior two problems and how the jumps to the left got smaller (ended closer to 6); have them consider the pattern for where the next jump would end.	Make connections to prior problems by having students think about the distance from the negative numbers to the positive ones.

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