2017

Preservice Teachers' Algebraic Reasoning and Symbol Use on a Multistep Fraction Word Problem

Amanda Miller
Illinois State University

Jennifer Tobias
Illinois State University

Elif Safak
Florida Gulf Coast University

J. Vince Kirwan
Kennesaw State University

Nicole Enzinger
George Fox University, nenzinger@georgefox.edu

See next page for additional authors

Authors
Amanda Miller, Jennifer Tobias, Elif Safak, J. Vince Kirwan, Nicole Enzinger, Megan Wickstrom, and Jae Baek

Preservice Teachers' Algebraic Reasoning and Symbol Use on a Multistep Fraction Word Problem

Amanda L. Cullen
Illinois State University
almille@ilstu.edu

Jennifer M. Tobias
Illinois State University
jtobias@ilstu.edu

Elif Safak
Florida Gulf Coast University
esafak@fgcu.edu

J. Vince Kirwan
Kennesaw State University
jkirwan@kennesaw.edu

Nicole M. Wessman-Enzinger
George Fox University
nenzinger@georgefox.edu

Megan H. Wickstrom
Montana State University
megan.wickstrom@montana.edu

Jae M. Baek
Illinois State University
jbaek@ilstu.edu

Previous research on preservice teachers' understanding of fractions and algebra has focused on one or the other. To extend this research, we examined 85 undergraduate elementary education majors and middle school mathematics education majors' solutions and solution paths (i.e., the ways or methods in which preservice teachers solve word problems) when combining fractions with algebra on a multistep word problem. In this article, we identify and describe common strategy clusters and approaches present in the preservice teachers' written work. Our results indicate that preservice teachers' understanding of algebra include arithmetic methods, proportions, and is related to their understanding of a whole.

Key words: Algebraic Reasoning, Fractions, Preservice Teacher Education

Research indicates that students' opportunities to learn are directly related to the tasks with which they engage (Cechlárová, Furcoňová, & Harminc, 2014; National Council of Teachers of Mathematics, 1991; Stein, Smith, Henningsen, & Silver, 2009). Situating problems in meaningful contexts often provide students with more opportunities to learn because they allow the problem solver to be creative in developing multiple solution paths (Lamon, 2007; Stacey & MacGregor, 1999). Researchers have analysed students' solution paths for a variety of purposes. For example, Carpenter, Fennema, Franke, Levi, and Empson (2015) found that word problems supported young children's thinking and reasoning about number and operations, including fractions.

Walkington, Sherman, and Petrosino (2012) asserted that word problems helped ninth grade students transition from arithmetic to algebraic thinking and provided them with a medium to explore fundamental mathematical ideas in algebra. Berk, Tabor, Gorowara, and Poetzel (2009) investigated elementary preservice teachers' flexibility in solution paths on missing-value proportion word problems (e.g., Nikita can paint 12 signs in 5 hours. How many signs can he paint in 25 hours?). They found that the participants exhibited a high level of

correctness yet limited flexibility in solution paths prior to exposure to one of two instructional interventions. After the instructional intervention, participants in both intervention groups exhibited an increase in awareness of multiple solution paths, an increased ability to solve the problem using multiple solution paths, and an ability to compare and contrast these different solution paths according to relative efficiency. These studies provide promising evidence that one's flexibility in using various strategies can be improved with experience and aid in transitioning students from concrete to more abstract ways of reasoning.

Several studies have demonstrated that teachers' mathematical content knowledge is a significant predictor of their students' mathematics achievement (e.g., Hill, Rowan, & Ball, 2005; Hill, Sleep, Lewis, & Ball, 2007). Thus, it is important for teachers to not only have a conceptual understanding of individual topics, but also connect and integrate their understanding of those topics (Wieman & Arbaugh, 2013). The purpose of this study was to investigate undergraduate elementary and middle school preservice teachers' (PSTs') solutions to a multistep word problem with fractions. In this article, we discuss the algebraic reasoning and symbolic representations present in their solution paths.

PSTs' Understanding of Algebra

Elementary, middle school, and secondary students worldwide have struggled with algebra content for years (C. Kieran, 1992, 2007). Research indicates that they have difficulties interpreting and operating on symbols representing unknowns (e.g., Hunter, 2010; Linchevski & Herscovics, 1996; MacGregor & Stacey, 1997), understanding the equal sign as a symbol indicating equivalence (e.g., Knuth, Stephens, McNeil, & Alibali, 2006), and representing situations symbolically (e.g., Hunter, 2010; Swafford & Langrall, 2000). Although what and how algebra should be taught in Kindergarten through Grade 12 (ages 5–17) is beyond the scope of this article, what is germane is how and what teachers think about algebra because research indicates that PSTs' learning experiences with algebra can affect their teaching behaviours (and thus learning outcomes; Cechlárová et al., 2014). For example, Van Dooren, Verschaffel, and Onghena (2002) found that PSTs' evaluations of their students' algebraic solutions were related to the PSTs' personal solutions. Based on this research, Van Dooren et al. concluded that PSTs need experiences developing, discussing, and critiquing multiple solution strategies as well as opportunities to reflect on their implicit privileging of arithmetic solutions in elementary school.

According to Stump and Bishop (2002), change can occur in as little as one semester. These researchers argued that their PSTs' definitions of algebra changed due to curricula materials and assignments in an algebra content course for elementary and middle school PSTs. Thus, "an important goal for mathematics teacher educators is to organise experiences for preservice teachers that will broaden their vision of algebra so that they can effectively promote the algebraic reasoning of elementary and middle school children" (Stump & Bishop, 2002, p. 1903). Yet the question remains—how should mathematics educators promote algebraic reasoning in methods and content courses? There are relatively few studies on elementary and middle school PSTs' understanding of algebra (Thanheiser, Browning, Edson, Kastberg, & Lo, 2013). Within this small research base, we have identified three main themes: PSTs' generalisations of patterns and the generation of explicit and recursive formulas (e.g., Hallagan, Rule, & Carlson, 2009; Richardson, Berenson, & Staley, 2009; Zazkis & Liljedahl,

2002), PSTs' understanding of the structure of algebraic expressions (e.g., Pomerantsev & Korosteleva, 2003), and PSTs' use and interpretation of mathematical objects, such as variables and the equal sign (e.g., Mills, 2012; Prediger, 2010). Furthermore, PSTs' difficulties with algebra, especially with their use of variables, may be linked to their understanding of symbolic representations (C. Kieren, 2007; Poon & Leung, 2010).

PSTs' Understanding of Fractions

Research illustrates that PSTs also have difficulties with defining referent wholes for fractions (Tobias, 2013) as well as understanding fractions as operators (Behr, Khoury, Harel, Post, & Lesh, 1997). When defining wholes for fractions, Tobias (2013) found that when asked to find how much of a pizza one person would get if four pizzas are shared equally among five people, only eight of 33 PSTs defined the correct whole as one pizza. In addition, research has shown that PSTs tend to favour partitioning strategies when asked to solve problems using a fraction as an operator (Behr et al., 1997). Behr, Khoury, Harel, Post, and Lesh (1997) found that when PSTs were asked to show 3/4 of 8 four-stick bundles, they tended to solve the problem by partitioning first, such as dividing the piles of sticks by four, as opposed to using measurement or stretching and shrinking methods to solve the problem, such as taking three groups of the 8 four-stick bundles.

Other research has shown that PSTs' difficulties with defining wholes and being flexible with fractions as operators is also prevalent in the context of fraction multiplication (Çağlayan & Olive, 2011; Luo, Lo, & Leu, 2011). Çağlayan and Olive (2011) asked PSTs to represent $\frac{1}{2} \times \frac{1}{3}$ using pattern blocks. Rather than representing 1/2 of 1/3, some PSTs represented 1/2 and 1/3 separately, showing 1/2 and 1/3 out of two separate wholes. Then, the PSTs placed a multiplication sign in the middle of the two representations. Similarly, Luo, Lo, and Leu (2011) found that in a study with PSTs from the United States and Taiwan that PSTs struggled to identify a correct picture for $\frac{3}{4} \times \frac{4}{5}$. When given choices, PSTs found the picture in Figure 1 to be incorrect when in fact it correctly represents $\frac{3}{4} \times \frac{4}{5}$. When asked why they thought that picture was incorrect, PSTs from both countries responded that they thought the fractions should be out of the same whole meaning that both fractions should be out of the whole rectangle. Thus, they selected the picture representing 3/4 of 4/5 as incorrect because the whole for 3/4 is 4/5 and the whole for 4/5 is the whole rectangle (see Figure 1).

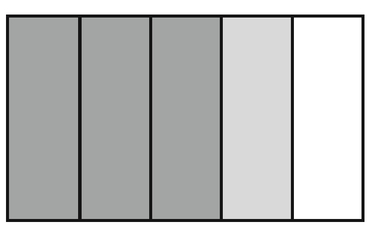

Figure 1. Correct representation for $\frac{3}{4} \times \frac{4}{5}$.

Linking Fractions and Algebra

Although PSTs have difficulties with the mathematical topics of fractions and algebra, often applying procedures and algorithms without understanding (Çağlayan & Olive, 2011; Kaput, 1998; C. Kieran, 2007; Luo et al., 2011), it is important for mathematics educators to link these topics. Research has highlighted that understanding meanings of fractions provides students with a foundation for understanding concepts and ideas in algebra (Behr, Lesh, Post, & Silver, 1983; T. E. Kieren, 1993; National Mathematics Advisory Panel, 2008). For example, understanding a fraction as an operator, or a fraction that transforms a number, not only "highlights certain algebraic properties,…but also provides experience with the notion of composite functions in a fairly concrete way" (T. E. Kieren, 1993, pp. 59–60). Similarly, partitioning units and being able to do so flexibly "allows the algebraic notions of operations and equivalence to emerge" (T. E. Kieren, 1976, p. 124). In addition, ideas with fractions and ratios provide students with a foundation for conceptualising slope (Tobias, 2009).

It is also important for PSTs to apply their integrated understanding of fractions and algebra in a problem-solving context, such as utilising fractions as quantities instead of only whole numbers in algebraic word problems. Word problems that include multiple topics are more cognitively demanding (cf. Stein et al., 2009) for students than those which focus on individual or isolated topics, and they also reveal students' conceptions about both topics as well as their connections. For example, solving a word problem that combines fractions and algebra necessitates a strong understanding of both topics and this understanding facilitates flexibility. Solving problems with multiple representations and making connections among them is important for students to "understand how mathematical ideas interconnect and build on one another to produce a coherent whole" (National Council of Teachers of Mathematics, 2000, p. 64) and to develop a deep understanding of a problem situation (Wieman & Arbaugh, 2013). This is important because different representations highlight different characteristics of the relationship under study (Friedlander & Tabach, 2001). Linsell and Anakin (2013) argued that modelling problem situations and using multiple representations are two of eight features of foundation content knowledge that PSTs need (pp. 444–445). Thus, mathematics educators must provide PSTs with opportunities to develop their ability to solve fraction and algebra problems with multiple methods and representations so that they can develop the content knowledge necessary to teach mathematics cohesively in their own classrooms.

Rationale for the Present Study

Previous research on PSTs' understanding of fractions and algebra has primarily focused on one topic or the other. The purpose of this study was to examine PSTs' solution paths (i.e., the ways or methods in which PSTs solve word problems) when combining fractions with algebra on a multistep word problem. This study was part of a larger study investigating the ways in which PSTs' understanding of fractions connects to their understanding of algebra by analysing how they solve the following word problem, hereafter referred to as the *Paycheck Problem*.

> Emily received her paycheck for the month. She spends 1/6 of it on food. She then spends 3/5 of what remains on her house payment. She spends 1/3 of what is then left on other bills. Finally, she spends 1/4 of the remaining money for entertainment. This activity leaves her with $150, that she puts into savings. What was her original take-home pay?

The PSTs were instructed to solve the problem pictorially and algebraically and to explain their work. In this article, we present a subset of the results by focusing on the PSTs' algebraic solutions. Our analysis of PSTs' pictorial solutions are discussed elsewhere (Baek et al., 2017). Therefore, the question that framed the research reported in this article is: *In what ways do PSTs incorporate symbolic representations in the solutions to the Paycheck Problem?*

We see the Paycheck Problem as a quality task that provided us with the opportunity to attend to multiple features of foundation content knowledge (Linsell & Anakin, 2013). Rather than posing a mathematically sequenced set of tasks prompting PSTs to solve one-step, then two-step equations, we required our PSTs to solve a multistep problem. It allowed us to investigate their understanding of the referent whole as well as multiplicative part-whole relationships—did they operate on multiple unknowns (changing wholes) or did they consider all parts (expenses) in terms of one unknown (the paycheck)? We could also examine how they represented the problem situation symbolically and how they connected multiple representations. Although some may argue that it is better to treat each in isolation, we argue that combining fractions with algebra on a multistep word problem prompted more natural responses (cf. practice or build on procedures recently reviewed or used) and offered opportunities for richer classroom discussions that emphasised conceptual understanding and mathematical reasoning over procedural skill (e.g., symbol manipulation).

Method

This study arose from our work teaching the first of three mathematics content courses for undergraduate elementary education majors and middle school mathematics education majors. One of the purposes of this course was to deepen PSTs' mathematical content knowledge through problem solving and reasoning and to increase their conceptual understanding of rational number operations and properties using various representations. In our teaching of this course, the PSTs routinely worked in groups of four to six as the mathematics educator circulated the room to ask questions with the purpose of extending the PSTs' thinking. After a set amount of time, PSTs were asked to present, explain, and justify their solution paths. Then, the mathematics educator facilitated a discussion on the presented solution paths, providing PSTs with opportunities to compare and contrast solution paths. Also during this discussion, the mathematics educator helped the PSTs connect the solution paths back to the mathematical goals of the task.

The participants (N = 85) were undergraduate elementary PSTs and middle school mathematics PSTs enrolled across five sections of the aforementioned mathematics content course at a university in the Midwestern region of the United States. These sections were taught by four of the authors of this article during the fall semester of 2013, but all authors have taught the course at some point. The content of this course focused on whole number and fraction concepts and operations. Approximately half of the course was spent on fractions.

Algebra was incorporated into the course when appropriate (e.g., starting with a word problem and writing an equation to represent the problem).

Our previous experience with PSTs enrolled in this mathematics content course informed us that the solutions PSTs provided differed greatly in various ways, including content, terminology, notation, and representations used. PSTs themselves noted that they struggled to solve the Paycheck Problem. This may have been due to the fact that Paycheck Problem was a novel word problem for the PSTs; it was the first time the PSTs were required to solve a multistep word problem in the course. However, based upon our discussions with mathematics educators at other universities, we do not see this problem as unique to our preparation program and university.

To examine and describe PSTs' symbolic representations in their solutions to the Paycheck Problem, we used a qualitative methodology. Our data consisted of the written work produced by the PSTs. We next discuss how we analysed the qualitative data that we collected.

Data Analysis

Our data analysis consisted of three sequential phases. In the first phase, the research team (i.e., the authors of this article) read through and sorted all of the PSTs' solutions into categories based upon the reasoning exhibited using the constant comparative method (Glaser & Strauss, 1967). Upon completion of this initial review, during the second phase, we met several times with an effort to establish strategy code definitions for the reasoning PSTs exhibited in their work. In the third phase, we selected a small subset of the data from one class to independently code the solutions presented in each PST's work based upon the codes developed in the second phase. Our unit of analysis was not each PST but each individual solution provided by the PST.

These coding phases were repeated for four iterative cycles with a small subset of the data: members of the research team independently coding a subset of data, meeting to discuss coding discrepancies, and then revising our codes and code definitions (Miles, Huberman, & Saldaña, 2014). With each cycle, our percent agreement improved, and we reached an acceptable percent agreement (98%) during our fifth cycle. After we redefined our codes, definitions, and coding procedure, each researcher independently coded the solutions in the whole data set based on the established definitions. Then, we met and resolved all coding discrepancies. This assigned each PST's solution (i.e., unit of analysis) an agreed upon collection of strategy codes (each solution was coded with at least two and no more than five codes). During our coding process, we noted when solutions included the use of variables and when they did not. We also coded for correctness as determined by the PST's final answer to the problem.

Strategy Code Definitions

In this article, we discuss seven of the most common strategies observed in the PSTs' solutions to the Paycheck Problem for which we created the codes: static whole, dynamic whole, build up, break down, picture reliant, fraction as operator, and proportion. We

developed these strategy codes and their definitions over time during the three coding phases as discussed above. Next we present our definitions for the seven strategies.

A solution in which a PST solved the task utilising the static whole strategy used the same whole consistently without changing the whole throughout the problem solving process. In contrast, we classified a solution in which a PST redefined and changed the whole at least once during the problem solving process as having a dynamic whole strategy.

Two other central strategies that permeated PST solutions were the break down strategy and the build up strategy**.** A solution in which a PST started with the original paycheck and operated on this quantity by removing portions paid for various expenses to determine the whole paycheck based on the given amount remaining ($150) is classified as utilising a break down strategy. When a PST started with the amount remaining at the end of the problem ($150) and operated on this amount to determine the original (whole) paycheck, we classified that solution as having a build up strategy. The build up strategy is comparable to Koedinger and Nathan's (2004) unwind strategy. However, the two are not synonymous. Based on their work with secondary students, Koedinger and Nathan described the unwind strategy as an informal strategy in which the students worked backwards by attending the last operation first and inverting each operation to determine the unknown start value without algebraic symbols. In contrast, the build up strategy in this study may or may not utilise variables.

When PSTs that utilised a picture reliant strategy in their solution, they made explicit references in their explanations of their symbolic representations to the pictures they drew for solving the task, or their solution could not be interpreted without the picture as a reference (e.g., an unexplainable $900 original paycheck without any work to support where it was derived from).

A solution in which PST used some form of fraction multiplication, either through repeated addition or through using fractions as scalars was coded as employing the fraction as operator strategy. For example, PSTs' solutions that incorporated repeated addition, such as writing $\$150\left(\frac{1}{5}\right)+\$150\left(\frac{1}{5}\right)+\$150\left(\frac{1}{5}\right)$ instead of $\$150\left(\frac{3}{5}\right)$, were coded as using a fractions as operator strategy because the 1/5 in this case was transforming the $150. Fractions were used as scalars when PSTs solved the problem by setting up equations such as $\$150=\frac{3}{4}x$ then determining that $\$50=\frac{1}{4}x$. In addition, other ways fractions were used as scalars were seen in solutions in which PSTs generated equations that involved fraction multiplication, such as $\frac{3}{5}\left(\frac{9}{18}x\right)$.

When a PST that set up a proportion within their solution to solve for an unknown (typically denoted with a variable), we classified that solution as employing a proportion. For example, the last sentence in the problem indicate that 3/4 of the remaining paycheck was $150. Thus, solutions in which PSTs wrote statements such as $\frac{3}{4}=\frac{150}{x}$ were coded as using a proportion.

Results

Twenty-four of the 85 PSTs provided more than one solution in their written work such that each solution presented different mathematical thinking about the Paycheck Problem;

thus, we identified 114 individual solutions in the PSTs' written work. After coding these individual solutions, we identified emergent strategy clusters, which we define to be collections or clusters of strategies that PSTs utilised to reason about the whole and parts of the whole paycheck. The primary clustering was based on the static or dynamic whole and build up or break down codes, but the other codes varied within these four strategy clusters, as we were looking to make sense of what these codes meant in terms of the larger picture. These four main strategy clusters were (a) static whole, build up; (b) static whole, break down; (c) dynamic whole, build up; and (d) dynamic whole, break down (see Figure 2).

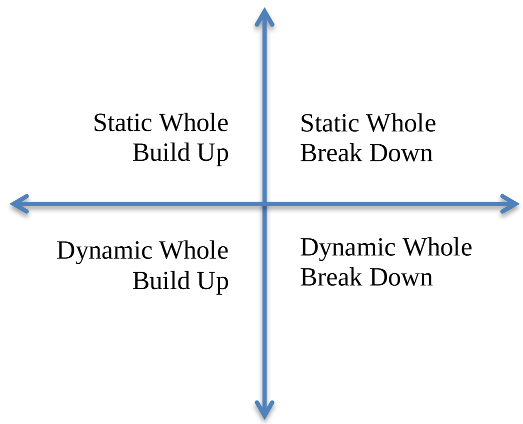

Figure 2. Four emergent strategy clusters.

Within these strategy clusters, we also identified approaches, which are collections of other codes or strategies that PSTs utilised to solve the problem. For example, within the static whole, break down strategy cluster, there were four approaches: (a) static whole, break down, and variables used; (b) static whole, break down, picture reliant, and no variables used; (c) static whole, break down, picture reliant, fraction as operator, and variables used; and (d) static whole, break down, fraction as operator, and variables used (see Table 1). In each of the following subsections, we first describe the strategy cluster and then present the two most common approaches utilising that particular strategy cluster. In Table 1 we report the distribution of each approach within each strategy cluster.

Table 1
Frequencies for Each Strategy Cluster and Approach

Strategy Cluster	Individual Solutions (*N* = 114)	PSTs (*N* = 85)	Number of correct solutions *n*	%[1]
Static Whole, Build up and Break Down	2	2		
Static Whole, Build up	40	34	23	57.5
Static Whole, Break Down	7	7	4	57.1
Dynamic Whole, Build up	38	34	29	76.3

Dynamic Whole, Break Down	27	21	13	48.1

[1]Percent correct for that strategy cluster.

Static Whole, Build up and Break Down Strategy Cluster

It is interesting to note that two PSTs incorporated a static whole, build up and break down strategy cluster. Both PSTs utilised a break down strategy first. For example, after finding the amount of the paycheck that was remaining after each step, Amy found that the final amount of the paycheck remaining was 1/6, which was equivalent to \$150 (see Figure 3). She then independently used a build up strategy to solve \$150 x 6 to determine that the total amount of the paycheck is \$900. Hence, we utilised a coordinate plane (see Figure 2) to represent our strategy clusters, allowing for these two PSTs' solutions to be included along the positive vertical axis.

$$\frac{6}{6} - \frac{1}{6} = \frac{5}{6} \text{ remaining}$$

$$\frac{2}{5} \cdot \frac{5}{6} = \frac{10}{30} = \frac{1}{3} \text{ remaining}$$

$$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \text{ remaining}$$

$$\frac{2}{9} \cdot \frac{3}{4} = \frac{6}{36} = \frac{1}{6} \text{ remaining}$$

$$\frac{1}{6} = \$150$$

$$150 \times 6 = \boxed{\$900}$$

Figure 3. Amy's solution utilising a static whole, build up and break down strategy cluster.

Static Whole, Build up Strategy Cluster

The most common strategy cluster used by the PSTs was static whole, build up. It was used in 40 out of 114 individual solutions (35%) by 34 PSTs across all classes (see Table 2). Within this strategy cluster, PSTs started with the amount remaining after paying all the bills, \$150, then worked backwards to determine the original paycheck (i.e., build up) without changing the whole (i.e., static whole). It was common for PSTs utilising this strategy cluster to consider the parts of the whole paycheck in terms of the whole paycheck. Of the 40 individual solutions utilising the static whole, build up strategy cluster, 23 were coded as correct (57.5%).

Table 2
Frequencies for the Static Whole, Build up Strategy Cluster

Strategy Cluster				Individual Solutions (N = 114)	PSTs (N = 85)
Static Whole, Build up				40	34
PR			NoV	17	16
PR	FO		NoV	8	8

	FO		NoV		5	5
			NoV		5	5
				V	3	3
PR	FO			V	1	1
PR		P		V	1	1

Note. PR = Picture Reliant, FO = Fraction as Operator, P = Proportion, NoV = No Variable, V = Variable.

Static whole, build up, picture reliant, no variable approach. The static whole, build up strategy cluster was employed within seven different approaches (Table 2). The most common approach within this strategy cluster was the static whole, build up, picture reliant, no variable approach (SW-BU-PR-NoV). Sixteen PSTs used this approach in 17 individual solutions. When employing this approach, PSTs used the picture to determine the amount of money spent on each expense before adding all of the expenses to determine the original paycheck amount. Hence, their algebraic solutions were reliant on their pictorial solutions. For example, Becca drew 18 circles and then drew diagonal segments in three of the circles to represent how much of the paycheck was spent on food, drew horizontal segments in nine circles to represent how much was spent on the house payment, heavily shaded in two circles to represent how much was spent on other bills, and lightly shaded one circle to represent how much was spent on entertainment (see Figure 4). The three unshaded circles were what remained after food, house payment, other bills, and entertainment and is equal to \$150. Hence, Becca determined that three circles represented \$150, so one circle represented \$50 and 18 circles represented \$900. Hence, Becca built up from one circle to 18 circles to determine the amount of the whole paycheck (i.e., 18 × 50). Because her whole was consistently 18 circles, her whole was static. She relied upon the picture to determine that because one circle represented \$50, 18 circles represented \$900, and therefore the original paycheck was \$900. Note that Becca did not use variables in her solution.

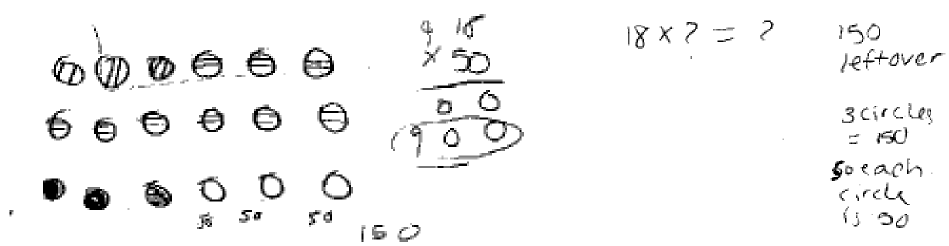

Figure 4. Becca's solution utilising a SW-BU-PR-NoV approach.

Static whole, build up, picture reliant, fraction as operator, no variable approach. The second most common approach within the static whole, build up strategy cluster was the static whole, build up, picture reliant, fraction as operator, no variable approach (SW-BU-PR-FO-NoV). Eight PSTs used this approach in one individual solution each. This approach was similar to the previous approach illustrated in Figure 4, with the addition of the use of fractions as operators. The use of fraction as operators can be seen in Figure 5. Although Troy wrote $\frac{3}{8} = \$150$, our team interpreted that Troy meant that 3/8 of the whole paycheck was

\$150. The scaling of the whole paycheck with fractions was why the additional code of fraction as operators was used.

Similar to Becca, Troy relied on his picture of 18 pieces to determine that three of 18 pieces of the paycheck represented \$150 (see Figure 5). However, Troy took it a step further to determine that 9/18 of the paycheck represented \$450, 2/18 of the paycheck represented \$100, and 1/18 of the paycheck represented \$50 and then added these parts, and thus build up from the parts, to compute the whole paycheck. Troy also did not use variables in his solution.

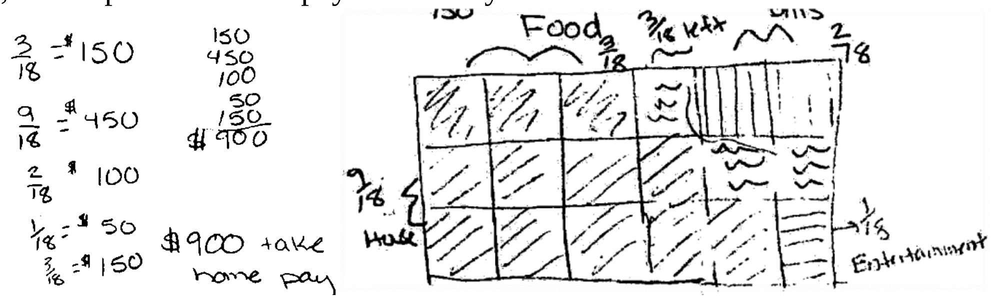

Figure 5. Troy's solution utilising a SW-BU-PR-FO-NoV approach.

Static Whole, Break Down Strategy Cluster

The least common strategy cluster used by the PSTs was static whole, break down. It was used in 6% of the 114 individual solutions. Seven PSTs from three of the classes used this strategy cluster in seven individual solutions (see Table 3). With this strategy cluster, PSTs started with the original paycheck and made payments (i.e., break down), usually one payment at a time, but did not consider the result of these payments until they were all made at once. Hence, they did not change the whole (i.e., static whole). Of the seven individual solutions utilising the static whole, break down strategy cluster, four were coded as correct (57.1%).

Table 3
Frequencies for the Static Whole, Break Down Strategy Cluster

Strategy Cluster					Individual Solutions (N = 114)	PSTs (N = 85)
Static Whole, Break Down					7	7
				V	3	3
PR			NoV		2	2
PR	FO			V	1	1
	FO			V	1	1

Static whole, break down, variable approach. The most common approach utilising this strategy cluster was the static whole, break down, variable (SW-BD-V) approach. Three PSTs utilised this approach in three unique solutions, and all three PSTs were inconsistent or incorrect in their use of variables. For example, Abby let *x* represent the original paycheck (see Figure 6).

$$x = \text{Paycheck}$$

$$x - \frac{1}{6} - \frac{3}{5} - \frac{1}{3} - \frac{1}{4} = 150$$

Figure 6. Abby's solution employing a SW-BD-V approach.

Then Abby subtracted each of the fractions reported in the problem (i.e., 1/6, 3/5, 1/3, 1/4), but she did not adjust the whole to show how each fraction introduced was indicating the fractional part of the *remaining* whole spent on each expenditure (i.e., static whole). Although Abby used a variable to represent the unknown initial paycheck amount, she wrote an incorrect equation and was unable to solve for *x*.

Static whole, break down, picture reliant, no variable approach. The next most common approach was the static whole, break down, picture reliant, no variable (SW-BD-PR-NoV) approach, but it was used by only two PSTs. Hailey used her picture (see Figure 7) to determine that the paycheck was \$900. Then she subtracted the amounts spent on each bill, writing "$900 - (150 + 450 + 100 + 50) = 150$." Hence, Hailey determined how much was spent on each bill by using the picture, which may have been one reason why Hailey was more successful than Abby.

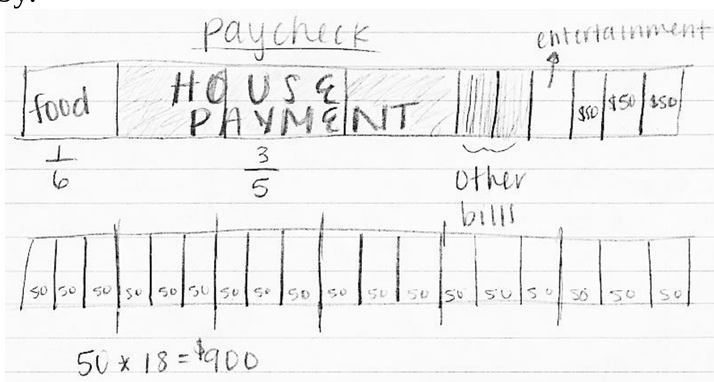

Figure 7. Hailey's solution utilising a SW-BD-PR-NoV approach.

Dynamic Whole, Build up Strategy Cluster

A third strategy cluster employed by the PSTs was the dynamic whole, build up strategy cluster. Thirty-four PSTs across all classes used this strategy cluster in 38 individual solutions (four PSTs used it in two solutions each; see Table 4). Hence, it was used in 33% of the 114 individual solutions. Within this strategy cluster, PSTs started with the amount remaining after paying all the bills, \$150. They then worked to determine the original paycheck (i.e., build up) by redefining the whole (i.e., dynamic whole) after accounting for each expense. Of the 38 individual solutions utilising the dynamic whole, build up strategy cluster, 29 were coded as correct (76.3%), which yielded the highest percent correct for the four strategy clusters.

Table 4
Frequencies for the Dynamic Whole, Build up Strategy Cluster

Strategy Cluster					Individual Solutions (N = 114)	PSTs (N = 85)
Dynamic Whole, Build up					38	34
	FO		NoV		23	22
		P		V	5	5
	FO	P		V	3	3
			NoV		3	3
PR	FO	P		V	1	1
PR	FO			V	1	1
PR	FO		NoV		1	1
	FO		NoV		1	1

Dynamic whole, build up, fraction as operator, no variable approach. In the dynamic whole, build up strategy cluster, the most common approach was the dynamic whole, build up, fraction as operator, no variable approach (DW-BU-FO-NoV). This was the most common approach overall across all of the strategy clusters. Twenty-two PSTs used this approach in 23 individual solutions (one PST used it in two solutions; see Table 4). The DW-BU-FO-NoV approach was used in 82% of the solutions within the dynamic whole, building up strategy cluster, which was 20% of all 114 individual solutions.

The PSTs who employed this approach used fractions as operators to scale each redefined whole to determine the original paycheck from the remaining \$150. However, because these PSTs did not use variables, many of their number sentences were mathematically incorrect. Eleven PSTs wrote incorrect number sentences to notate this approach, as illustrated by Kim's work in Figure 8. Kim started with the amount remaining after paying all the bills. This was 3/4 of the amount remaining after paying the "other bills" category of the problem. Hence, she wrote the number sentence $150 = \frac{3}{4}$. Although we speculate that the PST may have been computing $150 = \frac{3}{4}x$, where x is the amount of paycheck remaining after spending money on food, house payment, other bills, and entertainment, her number sentence written was incorrect.

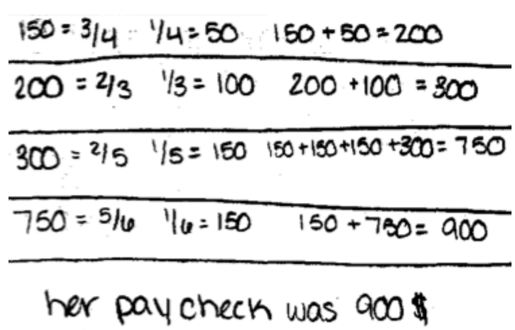

Figure 8. Kim's solution using a DW-BU-FO-NoV approach.

Kim then determined 1/4 of the paycheck remaining after spending money on food, house payment, other bills, and entertainment is \$50; hence, the new whole (amount of money from paycheck remaining before spending money on the entertainment bill) is \$200. Next, Kim continued this process to determine that \$200 is 2/3 of the amount of paycheck remaining after spending money on food, house payment, and other bills and identified a second new whole (amount of money from paycheck remaining before spending money on other bills) as \$300. Then, because \$300 is 2/5 of the amount of paycheck remaining after spending money on food and house payment, she identified a third new whole (amount of money from paycheck remaining before spending money on house payment) as \$750. Finally, Kim determined that \$750 is 5/6 of the amount of paycheck remaining after spending money on food, identified the fourth new whole as the original paycheck, and computed this to be \$900.

Dynamic whole, build up, proportion, variable approach. The second most common approach within the dynamic whole, build up strategy cluster was the dynamic whole, build up, proportion, variable approach (DW-BU-P-V). Five PSTs utilised this approach in five individual solutions. Carrie utilised this approach when she solved proportion equations to build up from the \$150 remaining after Emily paid all of her bills to determine (incorrectly) that her original take home pay was \$904. Although Carrie utilised *x* in each proportion, this *x* was the remaining paycheck before each bill was paid. Hence, Carrie was changing the whole in each proportion (see Figure 9).

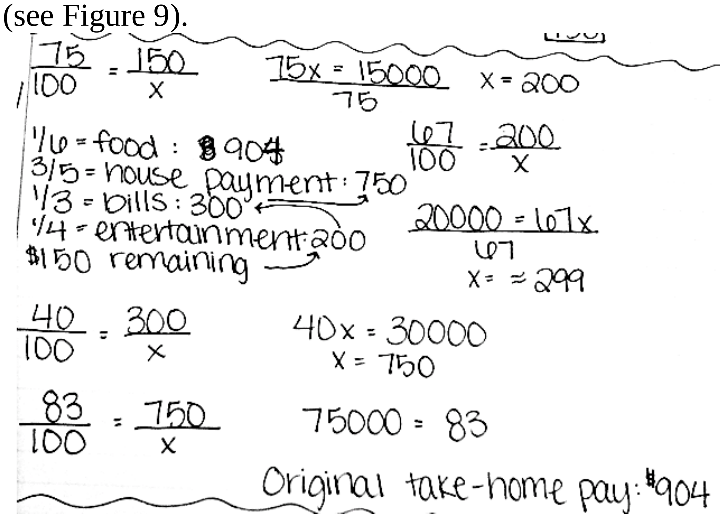

Figure 9. Carrie's solution employing a DW-BU-P-V approach.

Dynamic Whole, Break Down Strategy Cluster

As illustrated in Table 5, PSTs utilised the dynamic whole and break down strategy cluster in 27 out of 114 individual solutions (24%). PSTs utilised this approach by starting with the amount of the whole paycheck and then removing portions of it (i.e., break down) while accounting for the changing sizes of the remaining paycheck at different stages (i.e., dynamic whole) of the solving process. Of the 27 individual solutions utilising the dynamic whole, build up strategy cluster, 13 were coded as correct (48.1%), which yielded the smallest percent correct for the four strategy clusters.

Table 5
Frequencies for the Dynamic Whole, Break Down Strategy Cluster

Strategy Cluster					Individual Solutions (N = 114)	PSTs (N = 85)
Dynamic Whole, Break Down					27	21
PR	FO		NoV		8	6
PR	FO			V	8	7
	FO			V	6	6
PR			NoV		2	2
PR		P		V	1	1
	FO	P		V	1	1
				V	1	1

Dynamic whole, break down, picture reliant, fraction as operator, no variable approach. PSTs used seven approaches within the dynamic whole and break down strategy cluster. Two approaches were equally common within the dynamic whole, break down strategy cluster. One approach was the dynamic whole, break down, picture reliant, fraction as operator, no variable approach (DW-BD-PR-FO-NoV). Six PSTs used it in eight individual solutions (see Table 5). David's work illustrates this approach (see Figure 10). David began with $900 (i.e., the original paycheck), which was not stated in the original problem, indicating that David's solution of $900 came from somewhere else, which we later determined to be his pictorial solution. After determining the whole paycheck was $900, David determined 1/6 of $900 (i.e., $150), using the fraction 1/6 as an operator. David then subtracted $150 from $900 to obtain $750, the new whole. This removal or payment of $150 from the $900 paycheck to get $750, from which future removals or payments were then made was why this individual solution was classified as dynamic whole and break down. David continued this removal process by next determining 3/5 of $750 (i.e., $450) and subtracted this from the $750 to determine a second new whole. This process continued as he subtracted $450 from $750 to find a third new whole and calculated 1/3 of $300 (i.e., $100). Finally, David subtracted $100 from $300 to find a fourth new whole and determined 3/4 of $200 (but computed 3/4 of $200, or $150). This was the amount remaining after all the bills were paid. He recorded these steps in one number sentence (see the last line Figure 10).

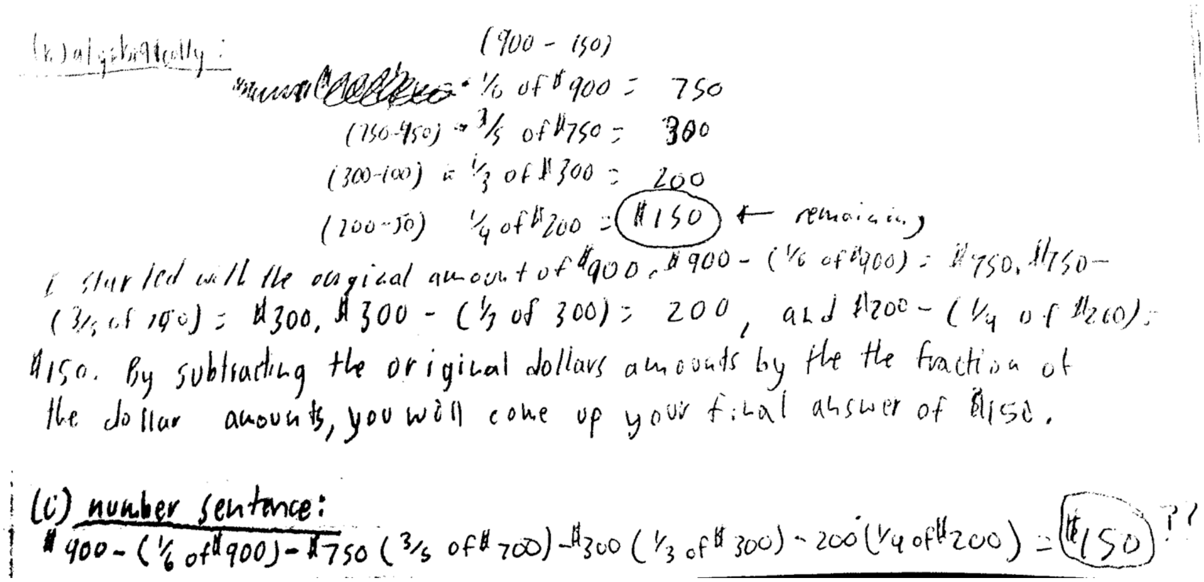

Figure 10. David’s solution utilising the DW-BD-PR-FO-NoV approach.

Dynamic whole, break down, picture reliant, fraction as operator, variable approach. The second of the two most common approaches was the dynamic whole, break down, picture reliant, fraction as operator, variable approach (DW-BD-PR-FO-V). It was used by seven PSTs in eight individual solutions (see Table 5). Katie’s work illustrates this approach (see Figure 11).

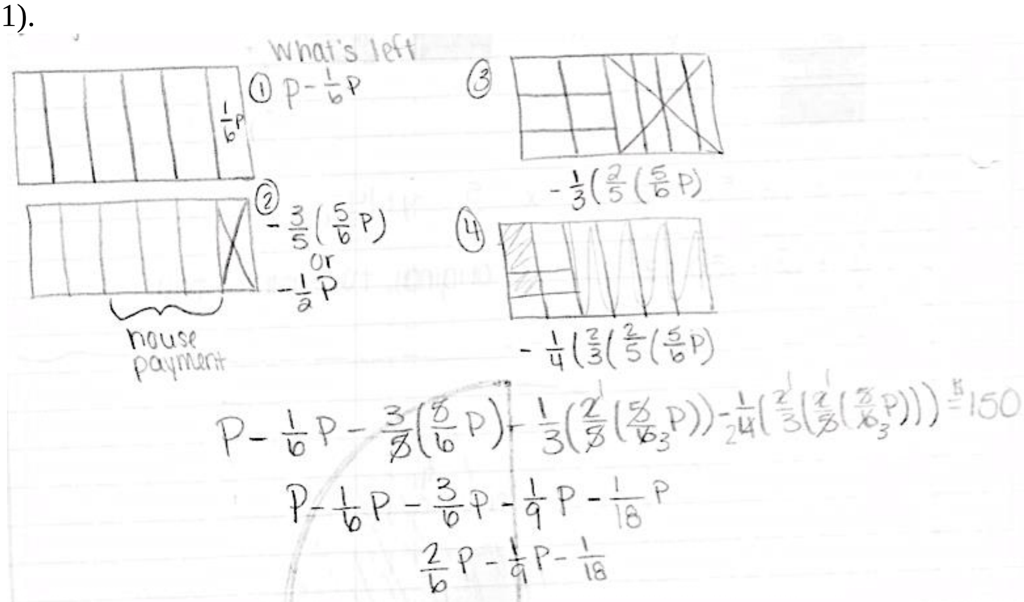

Figure 11. Katie’s solution employing the DW-BD-PR-FO-V approach.

Katie utilised the variable *P* to represent the original paycheck. Then, she made use of the changing wholes in stating that 3/5 of the remaining 5/6 of *P* must be spent on house payments. Katie continued to scale the paycheck throughout each step in her problem solving

by making payments and determining the new whole. Katie symbolised this scaling as a product of fractions. Katie provided her symbolic representation alongside her pictorial representation, demonstrating that her approach was picture reliant. Katie's approach differed from David's (see Figure 10) in how she symbolised the scaling of the paycheck by subtracting payments from a fixed amount to determine the amount of the paycheck remaining. Although both PSTs utilised fractions as operators, the component that distinguishes them is the use of a variable.

Discussion

In our investigation of the PSTs' solution paths on a multistep word problem, we identified numerous ways in how PSTs symbolised their solution paths. In this article, we discussed seven of the most common strategies (i.e., static whole, dynamic whole, build up, break down, picture reliant, fraction as operator, and proportions) in the form of four strategy clusters (i.e., static whole, build up; static whole, break down; dynamic whole, build up; dynamic whole, break down) and 28 approaches (e.g., static whole, break down, picture reliant, and no variables used approach).

We found that the static whole, build up strategy cluster was the most common strategy cluster, occurring in 40 out of 114 individual solutions. In contrast, the static whole, break down strategy cluster was the least common strategy cluster, occurring in only seven of the individual solutions. These trends with the strategy clusters may have been affected by PSTs' preference for how to operate on the whole and how to represent the problem context. More PSTs' solutions utilised a build up strategy over a break down strategy (78 vs. 34). This was expected considering that the problem implied a working backwards strategy with being given the remaining amount of the paycheck after all of the expenses. However, approximately one-third of the solutions included working from the beginning of the problem instead to find the original amount of the paycheck. Furthermore, approximately the same number of solutions utilised a dynamic whole, build up strategy cluster as a static whole, build up strategy (38 vs. 40). This was not expected. Because the Paycheck Problem was a multistep, start-value-unknown problem, we had conjectured that a dynamic whole, build up strategy cluster would be more accessible in accounting for each of the different steps when building up. In retrospect, our finding that approximately the same number of solutions utilising these two clusters is explainable in the PSTs' use of pictures. Twenty-seven of the solutions utilising a static whole, build up strategy cluster were picture reliant (i.e., 67.5%), yet only three of the solutions utilising a dynamic whole, build up strategy cluster were picture reliant (i.e., 8%). Hence, the PSTs' use of pictures affected their algebraic solution paths as this could have increased the number of PSTs that used a static whole build up strategy.

Very few PSTs' solutions included the use of proportions (12 out of 114) compared to those that used fractions as operators (71 of 114 solutions) with five of these PST solutions employing both. Every solution that included proportions also included variables, whereas only some of the fraction as operator solutions did.

Implications for Research

This study extends previous research that has shown that PSTs tend to not change the whole for fraction multiplication (Çağlayan & Olive, 2011; Luo et al., 2011) by illustrating that PSTs did this even when working within an algebraic task. Although 65 PST solutions utilised the dynamic whole strategy, only half of the 49 PSTs' solutions that utilised the static whole strategy indicated that they were aware of the fact that the whole was changing throughout the word problem. This is in contrast to research by Luo et al. (2011) that suggested that PSTs tend to keep the same whole throughout a fraction multiplication problem. Thus, incorporating multistep fraction problems may aid in PSTs' understanding of why and how wholes change for fraction multiplication. Further research is needed to support this claim.

Previous research suggests that word problems help to support young children's thinking and reasoning about number and operations including fractions (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 2015). Reasoning about fractions through a familiar context, a paycheck, helped PSTs make sense of the task. This is in contrast to previous research that has illustrated that PSTs struggle to make sense of problems (Luo et al., 2011). For example, Luo et al. (2011) posed a task that involved a fraction multiplied by fraction without a word problem. The present study illustrates that when given a word problem and asked to create a symbolic representation that supports solving the problem, the PSTs were fairly successful in doing so when they used a dynamic whole, build up strategy (76.3% of those solutions were coded as correct, see Table 1). Thus, our study supports the previous finding that presenting PSTs with word problems and asking them to solve them in multiple ways may support their development of different solutions methods. Future work is still needed to determine how PSTs' solution methods develop over time.

Our findings also suggest that many PSTs could not solve the problem algebraically independent of other methods. Our results illustrate that roughly half (53 of the 114 solutions) of the algebraic solutions were done so with the aid of a picture. This may have been because the problem specially asked for PSTs to solve the problem in both ways; however, the ways in which they used the picture to understand the algebra were not included in our data analysis. Thus, future work is needed to determine how PSTs coordinated their pictorial and algebraic solutions.

Implications for Teaching

This article provides mathematics educators with insight into the ways in which PSTs use symbols, which can inform and guide their instruction of PSTs. First, an explicit connection between symbol use and defining a whole for fractions may aid in PSTs' understanding of fraction operations. For example, focusing classroom discussions around understanding a fraction as an operator when wholes change can provide PSTs with an understanding of what it means to multiply fractions. Some of the PSTs did not identify the multiplicative part-whole relationships. For example, Abby (see her work in Figure 6) indicated that she knew there was an operation present but identified this exclusively as subtraction. Other PSTs, such as Amy, Troy, and Kim (see Figures 3, 5, and 8, respectively for their work) compartmentalised each of the steps into different unknowns rather than in relation to one unknown. Mathematics

educators can support PSTs' development of fraction operations by asking the PSTs to make it a mathematical norm to identify the referent (e.g., 150 is 3/4 of what? 200 is 2/3 of what?).

Second, some of the PSTs were successful with drawing a picture to represent the situation but struggled with linking this to a symbolic representation. Making explicit connections between different representations, such as pictorial and symbolic, during whole class discussions can deepen PSTs' numerical reasoning. For example, PSTs can present a pictorial representation simultaneously with a symbolic representation to highlight the similarities and differences between the two and determine why or why not the symbolic representation accurately represents the pictorial representation. PSTs can then also develop an understanding of which representations are appropriate for a situation and why, as well as develop understandings of defining wholes with variables when solving a problem algebraically. For example, the same variable can be used to represent the entire paycheck throughout the problem and PSTs can explore why only one variable is needed in this problem as opposed to using a different variable for every step. Mathematics educators can help PSTs connect representations by asking them to reflect on how each step is represented (cf. Linsell & Anakin, 2013).

Third, providing multiple entry points for a given problem can also aid PSTs with determining when solution methods include errors or misconceptions and why. For example, PSTs who could not solve the problem algebraically without the aid of a picture (e.g., Troy's work in Figure 5) illustrated that some of them thought that solving with a picture first, then representing the solution with arithmetic methods, constituted an algebraic solution. PSTs can then explore the two solution methods to one another to deepen their understanding of what it means to solve a problem algebraically as opposed to using arithmetic methods to arrive at the correct solution to the problem. This could also lead to conversations about different algebraic methods, such as working forwards versus working backwards to solve the problem, and how these methods, which are both valid strategies, compare to one another.

Fourth, most of the PSTs solved the problem without using a variable. In fact, only 32.4% of the PSTs' solutions incorporated variables. Research indicates that elementary and middle school students also struggle to represent situations symbolically (Swafford & Langrall, 2000). In light of Van Dooren et al.'s (2002) study which found that PSTs privileged arithmetic solutions when evaluating their students' work was related to the PSTs' personal solutions, our findings suggest that PSTs need more opportunities representing situations symbolically (cf. Hunter, 2010) as well as developing, discussing, and critiquing multiple solution strategies. From our study, we have learned that even the explicit request for PSTs to solve the problem pictorially and algebraically may not prompt all PSTs to represent the problem situation symbolically. Yet, by knowing this tendency, mathematics educators may more easily facilitate a classroom discussion on the pictorial and algebraic solutions within and among students.

Conclusion

Our results indicate that when asked to solve a problem algebraically, PSTs' solutions included arithmetic methods and proportions (algebraic reasoning) and may be related to their understanding of a whole (fractional reasoning). Our posing of a multistep problem and asking PSTs to solve it algebraically provided us with insight into their thinking of what it means to

solve a problem with algebra in the context of a fraction task. We found that most of the PSTs' solution paths were arithmetic in nature or picture reliant, but our results may not exhaustively account for all of our PSTs' fraction and algebra conceptions nor all solutions paths that PSTs in different courses or at other universities may provide. However, we assert that this study provides us with evidence that presenting problems that require symbolic representations as well as additional representations, such as pictures, can provide PSTs with methods to still be successful with solving a problem even if not all solution methods are successful.

References

Baek, J. M., Wickstrom, M. H., Tobias, J. M., Miller, A. L., Safak, E., Wessman-Enzinger, N., & Kirwan, J. V. (2017). Preservice teachers' pictorial strategies for a multistep multiplicative fraction problem. *Journal of Mathematical Behavior, 45*, 1–14. doi:10.1016/j.jmathb.2016.10.005

Behr, M. J., Khoury, H. A., Harel, G., Post, T., & Lesh, R. (1997). Conceptual analysis of preservice elementary school teachers' strategies on rational-number-as-operator task. *Journal for Research in Mathematics Education, 28*(1), 48–69.

Behr, M., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). New York, NY: Academic Press.

Berk, D., Tabor, S. B., Gorowara, C. C., & Poetzel, C. (2009). Developing prospective elementary teachers' flexibility in the domain of proportional reasoning. *Mathematical Thinking and Learning, 11*(3), 113–135.

Çağlayan, G., & Olive, J. (2011). Referential commutativity: Preservice K–8 teachers' visualization of fraction operations using pattern blocks. In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 303–311). Reno, NV.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Portsmouth, NH: Heinemann.

Cechlárová, K., Furcoňová, K., & Harminc, M. (2014). Strategies used for the solution of a nonroutine word problem: A comparison of secondary school pupils and pre-service mathematics teachers. *IM Preprint, series A, No. 4/2014.* Retrieved from http://umv.science.upjs.sk/phocadownload/userupload/gabriela.novakova/A4-2014.pdf

Friedlander, A., & Michal T. (2001). Promoting multiple representations in algebra. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics: 2001 yearbook* (pp. 173–185). Reston, VA: National Council of Teachers of Mathematics.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York, NY: Aldine.

Hallagan, J. E., Rule, A. C., & Carlson, L. F. (2009) Elementary school pre-service teachers' understandings of algebraic generalizations. *Montana Mathematics Enthusiast, 6*(1–2), 201–206.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406. doi:10.3102/00028312042002371

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 111–156). Charlotte, NC: Information Age.

Hunter, J. (2010). "You might say you're 9 years old but you're actually B years old because you're always getting older": Facilitating young students' understanding of variables. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 256–263). Fremantle, Australia: Mathematics Education Research Group of Australasia. Retrieved from https://www.merga.net.au/documents/MERGA33_HunterJ.pdf

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. In National Council of Teachers of Mathematics, Mathematical Sciences Education Board, & National Research Council (Eds.), *The nature and role of algebra in the K–14 curriculum* (pp. 25–26). Washington, DC: National Academy Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: Macmillan.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 707–762). Charlotte, NC: Information Age.

Kieren, T. E. (1976). *On the mathematical, cognitive and instructional foundations of rational numbers.* In R. Lesh & D. Bradbard (Eds.), Number and measurement: Papers from a research workshop (pp. 101–144). Columbus, OH.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Mahwah, NJ: Lawrence Erlbaum.

Knuth, E. J., Stephens, A. C., McNeil, M. N., & Alibali, M.W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*(2), 129–164.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629–667). Charlotte, NC: Information Age.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*(1), 39–65.

Linsell, C., & Anakin, M. (2013). Foundation content knowledge: What do pre-service teachers need to know? In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow (Proceedings of the 36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 442–449). Melbourne, Australia: Mathematics Education Research Group of Australasia. Retrieved from https://www.merga.net.au/documents/Linsell_et_al_MERGA36-2013.pdf

Luo, F., Lo, J.-J., & Leu, Y.-C. (2011). Fundamental fraction knowledge of preservice elementary teachers: A cross-national study in the United States and Taiwan. *School Science and Mathematics, 111*(4), 164–177.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics, 33,* 1–19.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: SAGE.

Mills, B. (2012). The equal sign: An operational tendency does not mean an operational view. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 745–748). Kalamazoo, MI.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel* (ED 00424P). Washington, DC: U.S. Department of Education.

Pomerantsev, L., & Korosteleva, O. (2003). Do prospective elementary and middle school teachers understand the structure of algebraic expressions? *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal, Vol. 1: Content Knowledge.* Retrieved from http://www.k-12prep.math.ttu.edu/journal/1.contentknowledge/pomer01/article.pdf

Poon, K.-K., & Leung, C.-K. (2010). Pilot study on algebra learning among junior secondary students. *International Journal of Mathematical Education in Science and Technology 41*(1), 49–62. doi:10.1080/00207390903236434

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13*(1), 73–93.

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers use of representation to reason algebraically. *Journal of Mathematical Behavior, 28*(2), 188–199.

Stacey, K., & MacGregor, M. (1999). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior*, *18*(2), 149–167. doi:10.1016/S0732-3123(99)00026-7

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). *Implementing standards-based mathematics instruction: A casebook for professional development* (2nd ed.). New York, NY: Teachers College Press.

Stump, S., & Bishop, J. (2002). Preservice elementary and middle school teachers' conceptions of algebra revealed through the use of exemplary curriculum materials. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Heide, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1903–1910). Athens, GA. Retrieved from http://files.eric.ed.gov/fulltext/ED471781.pdf

Swafford, J. O., & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*, 89–112.

Thanheiser, E., Browning, C., Edson, A. J., Kastberg, S., and Lo, J.-J. (2013, September 26). Building a knowledge base: Understanding prospective elementary school teachers' mathematical content knowledge. *International Journal for Mathematics Teaching and Learning.* Retrieved from http://www.cimt.plymouth.ac.uk/journal/thanheiser.pdf

Tobias, J. M. (2009). *Preservice elementary teachers' development of rational number understanding through the social perspective and the relationship among social and individual environments* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (UMI No. 3383698)

Tobias, J. M. (2013). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education, 16*(2), 85–103.

Van Dooren, W., Verschaffel, L., & Onghena, P. (2002). The impact of pre-service teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. *Journal for Research in Mathematics Education*, *33*(5), 319–351.

Walkington, C., Sherman, M., & Petrosino, A. (2012). "Playing the game" of story problems: Coordinating situation-based reasoning with algebraic representation. *Journal of Mathematical Behavior*, *31*(2), 174–195. doi:10.1016/j.jmathb.2011.12.009

Wieman, R., & Arbaugh, F. (2013). *Success from the start: Your first years teaching secondary mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Zazkis, R., & Liljedahl, P. (2002). Generalisation of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*(3), 379–402. doi:10.1023/A:1020291317178

Amanda L. Miller, Assistant Professor of Mathematics Education, Department of Mathematics, Illinois State University. Her research interests focus on preservice teachers' development of mathematical content knowledge as well as children's thinking and learning about geometric measurement concepts.

Jennifer M. Tobias, Associate Professor of Mathematics Education, Department of Mathematics, Illinois State University. Her research interests center on task design for content courses for prospective elementary teachers as well as developing prospective teachers' understanding of number concepts and operations.

Elif Safak, Assistant Professor of Mathematics Education, Department of Curriculum and Instruction, Florida Gulf Coast University. Her research interests focus on preservice teachers' development of mathematical content knowledge and learning as well as in service teachers' professional development.

J. Vince Kirwan, Assistant Professor of Mathematics Education, Department of Mathematics, Kennesaw State University. His research interests focus on preservice teacher's content knowledge related to algebra and preservice teacher preparation.

Nicole M. Wessman-Enzinger, Assistant Professor of Education, School of Education, George Fox University. Her research interests focus on children's and preservice teachers' thinking about number, and particularly, integers.

Megan H. Wickstrom, Assistant Professor of Mathematics Education, Department of Mathematical Sciences, Montana State University. Her research interests focus on teachers' perceptions and implementation of research into practice, preservice teachers' development of mathematical content knowledge, and the practice of mathematical modeling in the elementary classroom.

Jae M. Baek, Assistant Professor of Mathematics Education, Department of Mathematics, Illinois State University. Her research interests focus on children's and teachers' mathematical understanding in the domains of numbers, operations, and algebraic reasoning.