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Abstract

Previous research has documented that preservice teachers (PSTs) struggle with under- standing fraction concepts and operations, and misconceptions often stem from their understanding of the referent whole. This study expands research on PSTs’ understanding of wholes by investigating pictorial strategies that 85 PSTs constructed for a multistep fraction task in a multiplicative context. The results show that many PSTs were able to construct valid pictorial strategies, and the strategies were widely diverse with respect to how they made sense of an unknown referent whole of a fraction in multiple steps, how they represented the wholes in their drawings, in which order they did multiple steps, and which type of model they used (area or set). Based on their wide range of pictorial strategies, we discuss potential benefits of PSTs’ construction of their own representations for a word problem in developing problem solving skills.

1. Introduction

With the release of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and higher expectations for new teachers, it is more important than ever for preservice teachers (PST) to make sense of fractions beyond algorithmic operations. To become effective teachers, PSTs will need to understand mathematical content and be ready to support elementary students as they develop understandings of fractions beyond computational procedures, such as developing pictorial representations to represent fractions as well as connecting computational operations to story contexts.

PSTs often view fractions through a lens of numerous misconceptions and procedural rules (Graeber, Tirosh, & Glover, 1989; Simon, 1993). Algorithmic procedures often dominate learners’ reasoning and hinder their ability to develop conceptual understandings (Glass, 2004; Mack, 2000; Osana & Royea, 2011). It is also difficult for PSTs to conceptualize fractions and operate on them because reasoning about fractions is often in stark contrast to the procedural methods they were taught as elementary students (Osana & Royea, 2011).

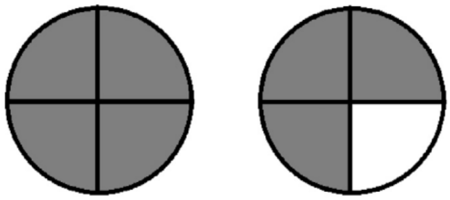

Fig. 1. "7/8 or 1 3/4? " Task to name a fraction that represents the shaded amount (from Tobias, 2013).

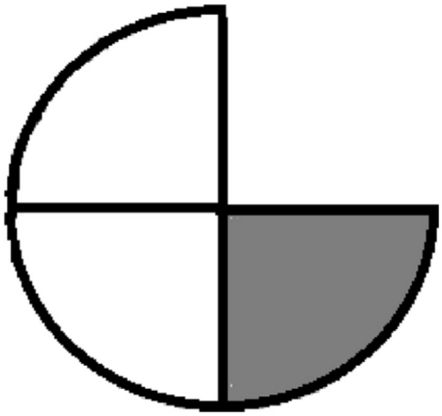

Fig. 2. "1/3 or 1/4?" Task to name a fraction that represents the shaded amount (from Tobias, 2013).

Particularly, it has been well documented that many PSTs do not understand the underpinning concepts of fraction operations such as fraction multiplication and division (Ball, 1990; Simon, 1993; Tirosh & Graeber, 1990).

More recently, several studies indicate that PSTs have difficulties with more fundamental concepts of fractions, such as understanding what the referent whole is for a given fraction (Luo, Lo, & Leu, 2011; Tobias, 2013). The studies by Luo, Lo, and Leu (2011) and Tobias (2013) indicate that PSTs need to clearly define the wholes of fractions before they operate on fractions, and a lack of clarity in defining wholes may be related to PSTs' confusion with fraction operations.

Although PSTs exhibit difficulties with fractions, multiple researchers highlighted that this is not always the case with elementary students (Mack 2001; Olive, 1999). Olive (1999) and Mack (2001) investigated how children utilize their knowl- edge of whole numbers, partitioning, and units, and reported that children could solve fraction problems in a multiplicative context in a way that makes sense to them and explain their method to others. In this study, we extend the research base on PSTs' understanding of referent wholes for fractions by examining the ways in which PSTs define multiple wholes through their valid and invalid pictorial strategies for a multistep word problem.

1.1. *PSTs' definition of fractional wholes and its relation to multiplicative computation*

Tobias (2013) examined discussions that arose while PSTs solved fraction tasks in which they were required to name a fraction for a shaded portion in a given picture. She documented that PSTs' discussions focused on determining the whole to which their fraction referred and language related to the meaning of the denominator. When Fig. 1 was presented to the class, the PSTs debated whether the shaded portion represented 7/8 or 1 3/4. The PSTs concluded that more clarification was needed when describing the fractions because a fraction that represents the shaded portion in the picture may be 1 3/4 or 7/8 depending on if the referred whole is one circle or two circles.

Tobias (2013) also documented that PSTs realized they needed to reference the whole when discussing a particular fraction. For example, when Fig. 2 was shown, they stated that the fraction could be 1/3 or 1/4. Tobias reported that when PSTs were asked how both could be a possibility, they realized they needed to define the referent whole (i.e., 1/3 of what?) to justify their reasoning.

With regard to operations on fractions, Luo et al. (2011) asked PSTs in the United States and Taiwan to select a pictorial representation that cannot be used to illustrate $3/4 \times 4/5$ or $4/5 \times 3/4$ (see Fig. 3). They found that most participants in both countries had difficulty with this task, and many selected the choice (a) as the incorrect representation. Through a follow-up discussion they found that PSTs

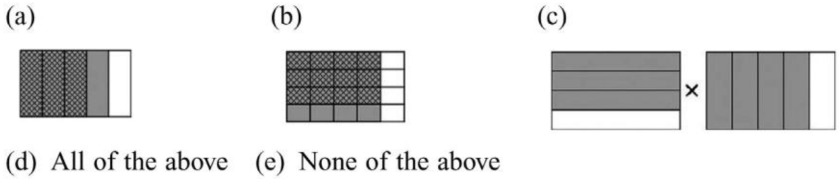

Fig. 3. Multiple choices for the task, "Which of the following pictures cannot be used to represent $3/4 \times 4/5$ or $4/5 \times 3/4$?" (from Luo et al., 2011).

chose the choice (a) as incorrect because they believed that the whole for 3/4 and 4/5 should be drawn to the same size and saw that the choice (a) had the whole for 3/4 to be smaller than the whole for 4/5.

Mack (2001) documented that fifth-grade students had similar confusions surrounding fractional wholes. She reported that students struggled with multiplying fractions when fractional wholes were not explicitly stated in word problems. For example, when students were given the following problem: "You have three fourths of a pizza. You give one third to a friend. How much pizza did you give your friend?," they were not sure if the problem was about $1/3 \times 3/4$ or $1/3 \times 1$. Mack discussed that when the problem clearly stated the whole for each fraction (e.g. three fourths of one whole pizza and one third of three fourth of the whole pizza), the students were able to explain each referent whole.

In these studies, the researchers documented that there is often confusion surrounding defining a whole and discussing fractions related to referent wholes. It is important for PSTs to sort out these conceptions and misconceptions because it affects their ability to determine and understand the meaning of operations as well as their ability to conceptualize situations involving fractions (Ball, 1990; Luo et al., 2011; Simon, 1993; Tobias, 2009).

1.2. *Supporting learning through pictorial representations for contextual problems*

One way to support PSTs in understanding rational numbers and operations on rational numbers is through facilitating their construction of strategies that make sense in a given context. With elementary students, Lamon (2007) documented that encouraging students to construct their own strategies for contextualized problems can help them develop deep conceptual understanding of fractions beyond traditional algorithms. Lamon (2007) stated, “children have tremendous capacity to create ingenious solutions when they are sufficiently challenged and when they do not feel expected to follow rules” (p. 653) and provided examples of contextualized “nontraditional” tasks that elicited student thinking (pp. 653–657). In addition, Empson and her colleagues (e.g., Empson & Levi, 2011; Empson, 1995) argued that encouraging elementary students to draw their own representations contributes to their understanding more than providing them with preformed fraction pieces in the long run because “to create workable representations, they need to reason about relationships such as how the number of parts is related to the whole unit” (Empson & Levi, 2011, p. 28).

In addition, Empson and Levi (2011) and Huinker (1998) emphasized that problems that are situated in meaningful contexts are important for students to make sense of fractions as quantities as well as to construct strategies for operations involving fractions. For example, Empson and Levi (2011) argued that sharing problems, such as 4 children sharing 5 candy bars, support students’ understanding of fractions as quantities, and that word problems, such as how many cookies fit on a whole tray if 15 cookies took up 3/4 tray (p. 213), support students’ understanding of multiplication and division involving fractions. Empson and Levi (2011) and Huinker (1998) discussed that strategies for such word problems can provide bases for developing numerical strategies and for solving equations with no contexts.

Furthermore, Mack (2001) discussed that fifth graders’ strategies for multiplying fractions were closely tied to the context of a given problem. For example, two problems involving $2/3 \times 3/4$ were perceived differently depending on if the problem was in the context of 2/3 of 3/4 of one whole pizza or if it was in the context of 3/4 of 2/3 of one whole pizza. Mack documented that elementary students were more readily able to solve the problems when they were in context a/b of b/c (e.g., 2/3 of 3/4) than b/c of a/b (e.g., 3/4 of 2/3) where $a < b$, $b < c$, $b \neq 0$, and $c \neq 0$). She argued that students could easily see two thirds within three fourths in the example of 2/3 of 3/4 of a whole pizza because the three fourths are already partitioned into three equal parts. Mack’s study indicates that problem contexts can influence the structure of multiplier and multiplicand in problems that involve two fractions, which, in turn, require two different types of reconceptualizing composite units. This suggests that contextualized problems can be more than introductory problems for students, and instead can be a carefully crafted instructional tool that facilitates students' learning of fractions. It is interesting to note that the discussions about the affordances of pictorial representations and contextual problems for students' learning of fractions are consistent with students' learning of the whole number domain (e.g., Carraher, Carraher, & Schliemann, 1985).

Even though researchers have documented elementary students' drawings for fractional word problems, little is known about PSTs' learning of fractional concepts through their pictorial strategies for similar tasks. There is a significant gap in the literature highlighting the types of representations or drawings PSTs produce and how these drawings support or hinder PSTs' understandings of fractions and related concepts. This study aims to extend this research by examining the multiple ways in which PSTs use pictorial representations and how these representations inform both conceptions and misconceptions of PSTs' understanding of the referent whole for fractions.

2. Methods

2.1 Context of the study and task

The data and results presented in this study were drawn from the first of the three mathematics content courses required for elementary and middle school PSTs. The broader purpose of the courses was for PSTs to problem solve, reason, and develop a deeper conceptual understanding of whole and rational numbers and operations. Each of the course sections was led with a reform-oriented instructional approach. During a typical class, the instructor posed mathematical problems, often multistep in context, to the PSTs. They were encouraged to construct multiple strategies and representations to solve the problem in small groups. The instructor acted as a facilitator so that PSTs could share their strategies, pose questions to one another, compare and contrast different strategies, and justify each strategy. Through this process, important concepts related to numbers and operations arose and were explored by the PSTs.

The authors of this study, each instructors of the course at some point in time, identified one problem, which we call the Paycheck problem, to be particularly revealing in terms of PSTs' understanding and misunderstanding of rational number concepts, operations, and representations. This problem read as follows, and PSTs were instructed to solve it pictorially and algebraically and to provide explanations:

Emily receives her paycheck for the month. She spends 1/6 of it on food. She *then spends 3/5 of what remains on her house payment. She spends 1/3 of what is then left for her other bills. Finally, she spends 1/4 of the remaining money for entertainment. This activity leaves her with $150, that she puts into savings. What was her original take-home pay?*

To solve this problem, PSTs often start by identifying the amount of the paycheck remaining after Emily spent 1/6 of it on food. The next steps require three fraction multiplication tasks and a task of relating the remaining $150 to the unknown initial paycheck. Of the three fraction multiplication tasks, two are about $a/b \times b/c$ if the PST solves the problem in the order given. For example, after Emily spends 1/6 of her paycheck on food, she is left with 5/6 of her paycheck. Because the remaining 5/6 of the paycheck is already partitioned into five equal parts, 3/5 would be readily identifiable within the remaining 5/6. In contrast, one step involving Emily spending 1/3 from a remaining 2/5 on her other bills requires partitioning the two equal parts in 2/5 into three equal parts. Mack (2001) reported that elementary students come to understand problems involving $a/b \times b/c$ before they are able to make sense of problem types involving $a/b \times c/d$ where $b \neq c$. The paycheck problem allowed us to investigate PSTs' understanding of fraction multiplication, partitioning, and pictorial representations that they construct in the processes because the problem involves both problem structures.

The paycheck problem was posed midway through the semester towards the beginning of a rational numbers unit, which was the second of the three units in this course. In the first unit, PSTs explored concepts related to place value, base 10, and alternative bases. In the second unit, PSTs explored problems related to understanding meanings of fractions, referent wholes, fraction equivalence, models to represent fractions (e.g., area, linear, and set models), and other rational numbers (e.g., decimals, percentages). In the second unit, prior to the paycheck problem, PSTs discussed tasks related to exploring referent wholes other than one. For example, they solved the following problems involving pattern blocks: "Charlie Brown takes two-fifths of the pattern blocks that Lucy has and gets a blue parallelogram, a yellow hexagon, and two green triangles. What pattern blocks might Lucy have had before Charlie took any away?"

PSTs also discussed strategies based on area and set models. For example, some PSTs solved the problem above using an area of one triangle as a unit (i.e., the area of Charlie's pieces are the equivalent to 10 triangles, and 10 was 2/5 of Lucy's pieces, so Lucy must have the pieces that have the area equivalent to 25 triangles). Others solved it using a set model (e.g., Charlie has 4 pieces, and 4 pieces were 2/5 of Lucy's pieces, so Lucy must have 10 pieces in total).

The Paycheck problem was the first problem in this course in which PSTs were asked to consider a fraction problem involving multiple steps, multiple referent wholes, and to solve the problem both pictorially and algebraically. The PSTs had not encountered or discussed a similar problem before. In teaching the same course in past, instructors noted that the paycheck problem provided a rich context, which often elicited discussions surrounding their understanding of fraction, meaning of fraction multiplication and division, definition of referent wholes, multiple strategies, and representations. This led us to formally investigate how PSTs construct pictorial and algebraic strategies, explain each strategy, and make connections between the strategies. In this article, we focus on the findings from PSTs' pictorial strategies of the problem.

2.2 Participants

During the fall semester of 2013, 85 PSTs from a public university in the Midwest participated in this study. The participants were elementary or middle level PSTs enrolled in one of the five sections of the mathematics content course described above. There were 130 PSTs enrolled in the five sections of this content course in total. All PSTs who agreed to participate in the study were included. The PSTs who volunteered for the study were similar to the total population in terms of their final grade distribution. Without any prior discussion of the problems or strategies, PSTs were given the paycheck problem described above, asked to solve it pictorially and algebraically, and directed to write explanations for each strategy. The instructors collected individual PSTs' written work, and we used scanned copies of their pictorial strategies and explanations as the data for the analyses in this study.

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Table 1
Number of strategies in each valid strategy category ($n = 75$).

Model	Working Forwards ($n = 67$)		Working Backwards ($n = 8$)	
	Singular whole ($n = 52$)	Multiple wholes ($n = 15$)	Singular whole ($n = 0$)	Multiple wholes ($n = 8$)
Area	47	11	0	7
Set	5	1	0	1
Combination	0	3	0	0

2.3 Data analysis

For this paper we focused on PSTs' pictorial strategies. A pictorial strategy was defined as one drawing with a PST's explanation for the referred drawing, if any. Some PSTs constructed multiple drawings with explanations, so our unit of analysis became a pictorial strategy rather than a participant. Of the 85 participants, six PSTs provided no pictorial strategy, nine provided two pictorial strategies, one provided three pictorial strategies, and one provided four pictorial strategies, which add up to 93 pictorial strategies in total.

We first analyzed PST's strategies for correctness, which we labeled as valid or invalid. Then, we classified the valid strategies into categories based on the following three factors: first, if the drawing included all the steps in one picture or a step for each expense was represented in separate pictures; second, if the pictorial representation was drawn starting with the unknown whole or starting with the last known dollar amount; and third, if the PST represented the given fraction based on an area model, a set model, or a combination of area and set models. The authors divided the 93 strategies and independently classified them. After the initial classification was completed, 20 of 93 strategies were double-coded. We had 85% agreement with minor discrepancies. After resolving the discrepancies, the rest of the data set was double- coded as well, and the authors agreed to the classification for each strategy. Strategies in each category are described in greater detail in the results section below.

3. Results

In this section, we describe different types of valid and invalid strategies using examples and explanations that the PSTs in the study provided. Of the 93 pictorial strategies, 75 were valid and 18 were invalid.

3.1 Different types of valid strategies

When considering valid strategies, we first classified the strategies by the way the problem was worked: working for- wards versus working backwards. Eighty-nine percent of the valid strategies (67 of 75 strategies) started with a pictorial representation of the unknown total paycheck and then represented each expense and remaining amount in the order of how it was stated in the problem, which we classified as working forwards strategy (see Table 1). This type of strategy was distinctly different from the working backwards strategy exhibited in the other 11% of the valid strategies. In the working backwards strategy, the PST initially represented the final remaining amount of $150 after all the expenses and then added on each expense in reverse order of how it was stated in the problem.

Next, we coded PSTs' strategies by the way they represented the whole paycheck: singular whole versus multiple wholes. In the singular whole strategy, the PSTs drew one polygon or one set of polygons to represent the amount of the total paycheck and represented all of the expenses within the original polygon(s). In the multiple wholes strategy, the PSTs drew one polygon or one set of polygons to represent the amount of the total paycheck and shaded the first expense, then drew another polygon(s) to represent the remaining amount and another expense, and continued the process to represent each remaining amount and subsequent expense. In this multiple wholes strategy, the PSTs defined a new whole after each expense.

Lastly, we coded PSTs' strategies by the models they employed in their drawings: area model, set model, or combination of area and set models. Table 1 shows the frequencies of PSTs' use of these different types of strategies. In the following section, we describe each type of strategies in detail using strategy examples and explanations that the PSTs provided.

3.1.1 Working forwards strategies

In this section, we describe different types of working forwards strategies, which the PSTs used most often. Through examples of the PSTs' work, we highlight differences between singular wholes and multiple wholes strategies as well as implementation of area, set, or combination models.

Fifty-two of the 67 working forwards strategies (78%) represented all the expenses as parts of the whole paycheck, which we classified as singular whole. Of the 52 strategies that used a singular whole, 47 strategies (87%) were based on the area model. For example, Wendy[1] first represented the whole paycheck

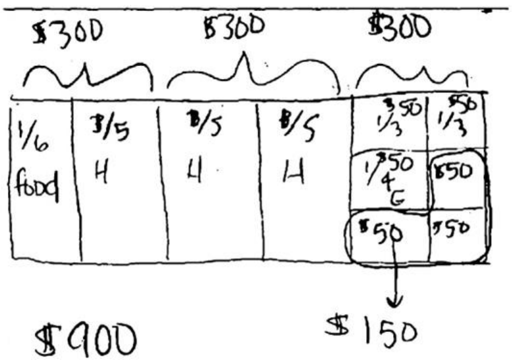

Fig. 4. Wendy's working forwards strategy using a singular whole based on an area model.

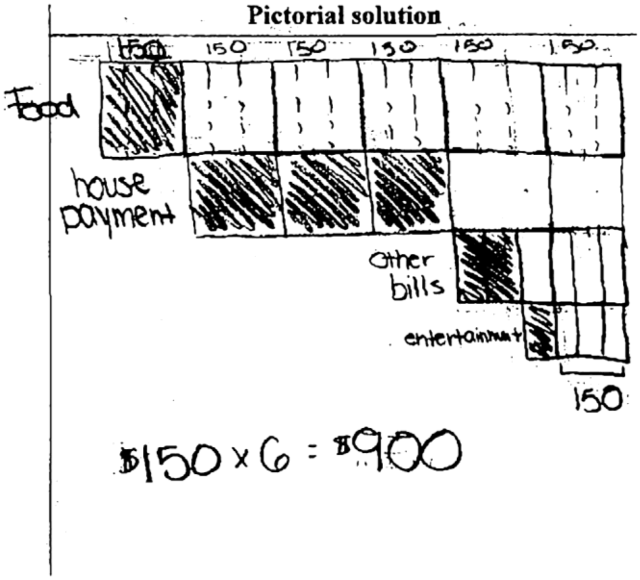

Fig. 5. Laura's working forwards strategy using multiple wholes based on an area model.

as one rectangle (see Fig. 4). She vertically partitioned it into six equal parts and labeled one of them "1/6 food." She then labeled each of the three equal parts "1/5 H," indicating 3/5 of the remaining was for the housing expense. Wendy further partitioned the two remaining parts, which resulted in six equal parts. She marked two of them "1/3" to notate the expense on other bills. Of the four remaining equal parts, she marked one of them "E" for the entertainment expense. She then circled the three remaining rectangles, marked them "$150," and wrote "$50" in each rectangle. She then marked each 2/6 of the total "$300," and figured out that the total paycheck was $900. Wendy's strategy was coded as working forwards because she represented the whole paycheck first and then partitioned out amounts based on payments described in the problem. It was coded as singular whole, because she drew one whole to represent what was occurring within the problem.

Eleven of the 58 working forwards strategies represented each remainder after each expense in a separate picture, which we classified as multiple wholes. For example, Laura drew a rectangle to represent the whole paycheck, vertically partitioned the rectangle into six equal parts, and shaded one of them for "food" (see Fig. 5). Then she redrew the remaining five rectangles underneath and shaded three of them, which she identified as the "house payment." She repeated this process of redrawing the remainders and shading expenses until she had three narrow rectangles representing "$150." She then went back to the first large rectangle and added in dotted lines to show the whole rectangle in the same sized parts as in the last rectangle. She

1 All student names in this article are pseudonyms.

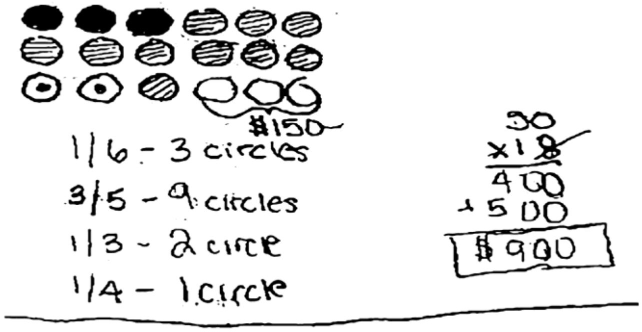

Fig.6. Tami’s working forwards strategy using a singular whole based on a set model.
(Explanation) To begin, I drew six circles to try and divide 1/6 of the money on food. As I continued to divide we realized six circles was not going to be enough. Therefore, I tripled the amount of circles to attempt to divide 18 circles. 1/6 of 18 circles is three. With three circles gone we are left with 15. 3/5 of 15 is 9 circles. With 9 circles gone we were left with 6 circles. An additional 1/3 was spent on bills and took up 2 more circles. Left with 4 circles, I shaded in 1 to represent 1/4 for entertainment. The remaining is three circles, which is equal to 150.

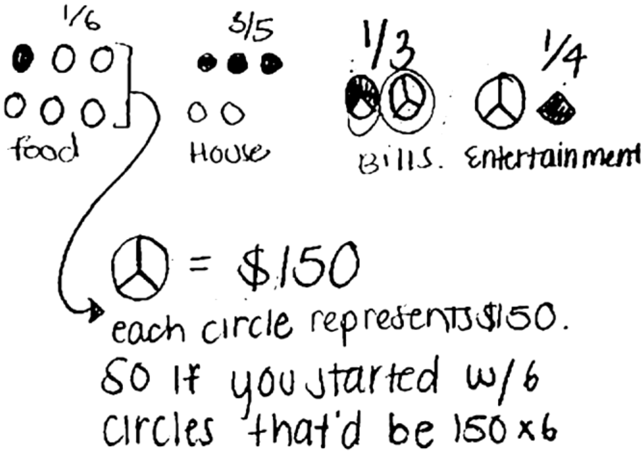

Fig. 7. Jamie’s working forwards strategy using multiple wholes based on a combination model.

computed “$150 × 6” to answer the question in the problem. Laura’s strategy exhibits a multiple wholes strategy because her drawing indicates that the portion of money left after each expense is a different amount.

In addition to the difference that Wendy used a singular whole to represent the total paycheck and all the expenses whereas Laura used multiple wholes, it is interesting to note that Wendy partitioned the rectangle horizontally and vertically, compared to Laura’s vertical partitioning throughout. The strategies by Wendy and Laura are similar in that they both used the size of the rectangle(s) to represent fractional parts of the dollar amounts in the problem.

Of the 67 valid working forwards strategies, six strategies (9%) were based on a set model. Five of the six strategies in this category were classified as using a singular whole because the strategies included a set of polygons to represent the whole paycheck as their first step, and each expense and corresponding remainder were presented in that set. For example, Tami’s strategy in Fig. 6 shows that she used 18 circles to represent the whole paycheck. Her explanation indicates that she initially drew circles and shaded one circle for food and three circles for housing. She then realized (see explanation below) that two circles were not enough to represent 1/3 of the remaining for other bills. She then drew 12 more circles underneath the original 6 circles. Tami used the 18 circles to re-represent each expense. After she figured out that three circles represent the final remaining amount of $150, she then knew each circle represents $50 and multiplied 50 by 18 to generate her answer, $900. Tami's strategy represented a working forwards, singular whole based on a set model strategy because she represented the whole paycheck as 18 distinct circles, which represented the whole, and then shaded in circles to represent the portions of the paycheck as they were specified in the problem.

Of the 67 working forwards strategies, three strategies (4%) were based on both area and set models. Jamie's strategy in Fig. 7 exemplifies this combination model. Jamie started with six circles to represent the whole paycheck and shaded one for food. She redrew the five leftover circles and shaded three for the house payment. She then redrew two remaining circles, partitioned each circle into thirds, and shaded 2/3 of one of the circles for bills. In her last picture, she redrew one circle and

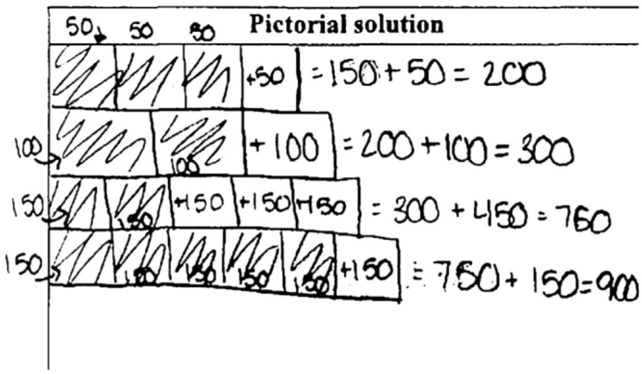

Fig. 8. Kim's working backwards strategy and explanation using multiple wholes.
(Explanation) You know she had $150 at the end and that was ¾ of what she had before entertainment. You need to add another $50 to $150 to make $200. $200 is actually 2/3 of what she had before her bills and since 2/3 equals #200. 1/3 would be $150 because $300 divided by 2 is $150. Since she spent 3/5 which is $300. $300 is 2/5 of what she had before her house payment. If $300 is 2/5 then 1/5 would be $150 because $300 divided by 2 is $150. Since she spent 3/5 of what she had and had 2/5 left over, you need to add 3 (sets of) $150 s to $300, so $300+$150+$150+$150=$750. $750 is 5/6 of what she had before food. $750 broken down is equal to $150, so you need to add on another $150 to make $900.

1/3 of the other circle and then shaded the 1/3 circle for entertainment. This left her with one circle, which represents the remaining $150. She then computed 150 × 6 to figure out her answer. We classified this type of strategies as combination of area and set models because the PST started with a set of six circles representing a quantity and then switched to 1/3 area of a circle representing another quantity. This strategy was also classified as multiple wholes because Jamie redrew a corresponding remainder after each expense.

3.1.2. Working backwards strategies

Of the 75 valid strategies, we classified eight strategies (11%) as the working backwards strategy. Within this strategy, the PST started with the last remaining $150 and added each expense to the corresponding remainder. All of the strategies in this working backwards category utilized multiple whole representations based on either an area model or a set model (see Table 1). Kim's pictorial strategy and explanation shown in Fig. 8 provides a window into her reasoning. Kim began by drawing a rectangle divided into fourths. She then identified in the last step that 1/4 of the remaining paycheck went to entertainment, so the $150 remaining represented 3/4 of the money that Emily had prior to paying for entertainment. Because $150 represented 3/4 of what was left prior to entertainment expenses, Kim figured out that the entertainment expense must be $50, which led to the conclusion that Emily must have had $200 dollars prior to paying for entertainment. Kim drew another rectangle underneath, divided it into three equal parts, and shaded two of them to represent $200, which was 2/3 of the money that Emily had left prior to pay for other bills. She then reasoned that 1/3 of it is $100. Kim continued this process until she determined Emily's whole paycheck.

Although she reasoned that the four rectangles representing $200 in the first row is 2/3 of the rectangles in the second row, it is interesting to note that Kim did not draw same sized rectangles to represent the same amount of money. In Kim's strategy, the area of three rectangles representing $200 in the first row did not match the two rectangles representing $200 in the second row, and this mismatch continued in the rest of the rows. Similar mismatched areas were observed in six of the eight strategies in the working backwards category. It is not clear whether these PSTs thought that each polygon represented a separate step in the problem, and therefore they did not see a need to represent the same amount using same size polygons.

3.2. Invalid strategies and conceptual struggles

Of the 93 strategies collected, 18 strategies (19%) were invalid or incorrect. Fifteen of the 18 strategies were invalid due to misconceptions. Of the remaining three strategies, two strategies were incomplete, and one included a computational error. In this section, we focus on the 15 invalid strategies related to misconceptions. Similar to the valid strategies, we examined if PSTs' invalid strategies were classified as working forwards or backwards, used a singular whole or multiple wholes, and were based on an area model, a set model, or a combination. Their strategies revealed that most misconceptions occurred when PSTs used a working forward strategy with an area model (see Table 2).

We identified three types of misconceptions or difficulties that are related to the 15 invalid strategies: understanding changing referent wholes, coordinating relationships between different size parts, and connecting pictorial representations with whole number quantities, The analyses revealed that six invalid strategies were related to the first type of misconception, two invalid strategies were related to the second type, five invalid strategies were related to the third type, and two

Table 2
Number of invalid strategies in each category ($n = 15$).

Model	Working Forwards		Working Backwards	
	Singular whole	Multiple wholes	Singular whole	Multiple wholes
Area	5	8	0	1
Set	0	0	0	1
Combination	0	0	0	0

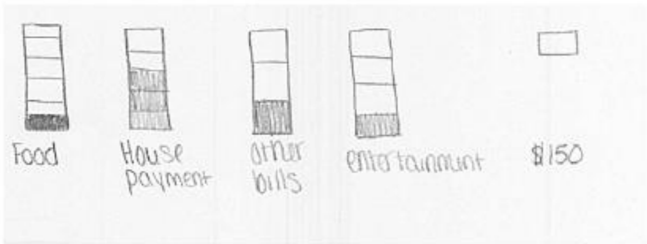

Fig. 9. Katie's invalid strategy: using the same size wholes.

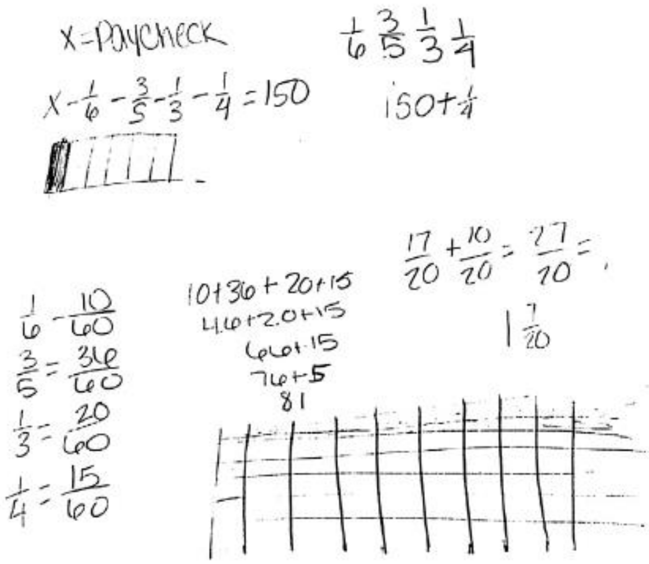

Fig. 10. Abby's invalid strategy: adding fractions with different referent wholes.

invalid strategies were related to both second and third. In this section, we describe each type with examples of the invalid strategies.

3.2.1. Understanding changing wholes

Within the invalid strategies, many PSTs found difficulty conceptualizing that each expense was a fractional part of a different sized whole. Katie's strategy in Fig. 9 exemplifies this struggle. She drew rectangles in which each of the four expenses was from the same sized wholes and could not determine the paycheck based on her drawing.

This idea of same sized wholes was an underlying misconception for five other invalid strategies in which PSTs found a common denominator for all fractions in the problem, indicating that they were thinking that the given fractions should be added or subtracted to find the original paycheck. As Abby's strategy in Fig. 10 exemplifies, when the PSTs computed the sum of the fractions, it was greater than one, and they were unsure of where to go from there. After determining the sum of all the expenses, represented as the mixed number, 1 and 7/20, Abby also drew a rectangle and partitioned it into 6 by 10 grid to represent 10/60. She shaded 1/6 of the 6 by 10 grid (i.e., one row) to indicate 1/6 of the paycheck was used for food, but then she did not use the picture in generating her answer 17/20.

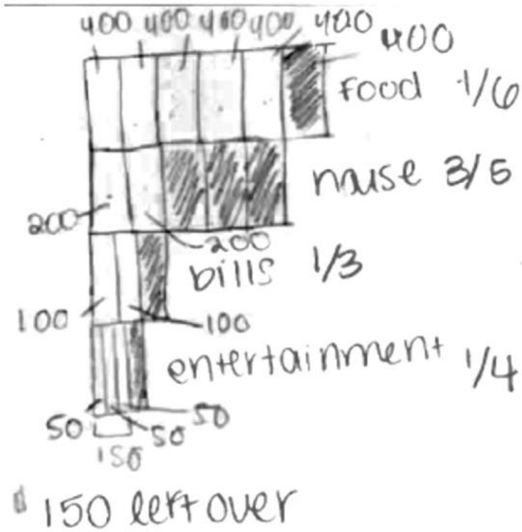

Fig. 11. Lizzie's invalid strategy: incorrectly representing 1/3 of 2/5

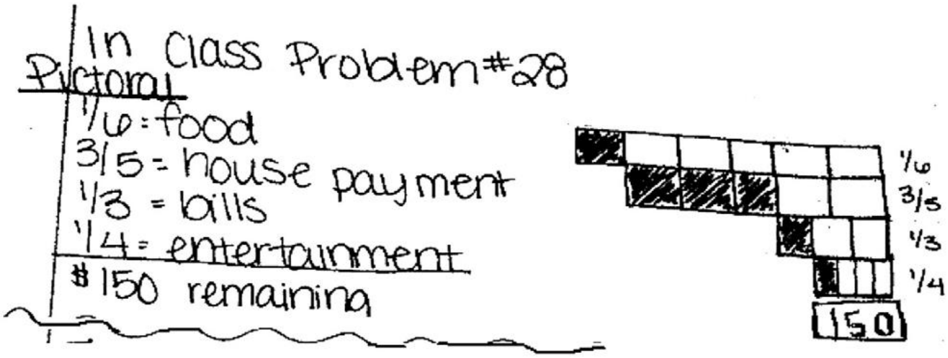

Fig. 12. Carrie's invalid strategy: not connecting pictorial representation to whole numbers.

3.2.2. Coordinating different sized parts

Another difficulty observed in PSTs' invalid strategies was related to how to coordinate the number of equal parts and certain fractional operators. After correctly representing 1/6 and 3/5 of the respective remaining paycheck, two PSTs struggled with partitioning the remaining two sections into thirds to indicate the 1/3 of what was remaining for other bills. For example, Lizzie's multiple wholes strategy in Fig. 11 shows that she partitioned a rectangle into six equal parts and labeled one of the six equal parts "Food." In the second row, she redrew the five remaining parts and shaded the three of the five equal parts "house," which left her with two equal parts left. When she needed to partition two remaining parts into three equal parts to represent "bills," Lizzie incorrectly represented three parts in the third row by drawing two of the three parts in the same size as the last third part. Although she correctly identified $50 as the amount for entertainment, her struggle of representing 1/3 of 2/6 of the total paycheck led Lizzie to incorrectly determining that each 1/6 of the paycheck was $200.

3.2.3. Connecting pictorial representation with whole number quantities

Five PSTs struggled to connect pictorial representation of fractional operators to whole number quantities. Two PSTs, including Carrie (see Fig. 12), pictorially represented each expense and the leftover amount of $150 correctly but did not make the connection between fractional parts and the remaining whole number dollar amount $150.

Connecting the pictorial representation with whole number quantities was an underlying challenge for three PSTs. They successfully identified the amount in their pictorial representation in the first few steps but made errors in later steps. For example, Tony's strategy in Fig. 13 shows that he correctly represented changing wholes in the first three steps of his working backwards strategy, but he made errors in the last step. More specifically, Tony first figured out that each fourth was $50 when 1/4 was spent on entertainment with $150 left. In the second row, he represented 1/3 for the other bills expense in one

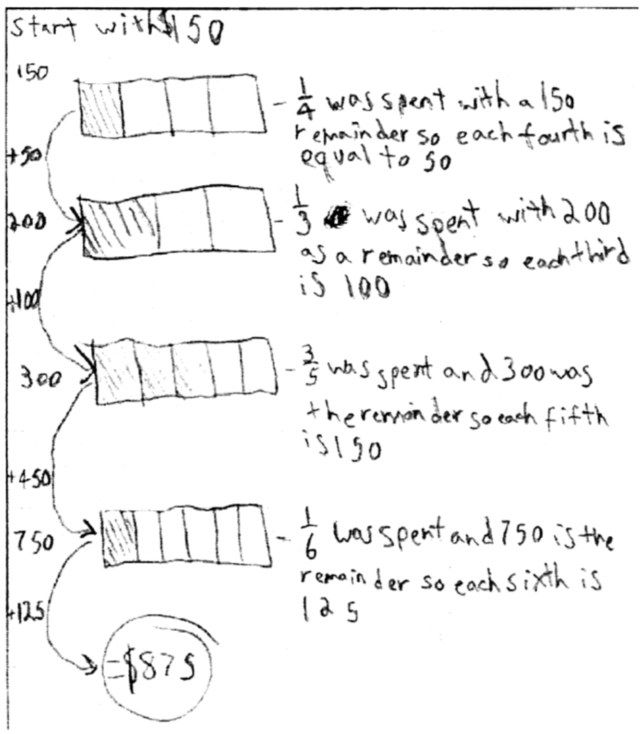

Fig. 13. *Tony's invalid strategy: partially connecting pictorial representation to whole numbers.*

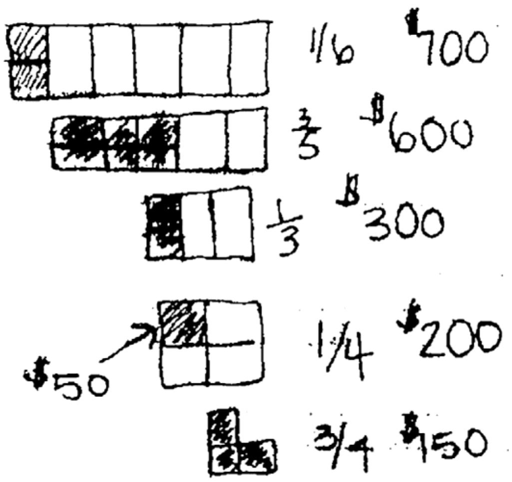

Fig. 14. Rachel's invalid strategy.

unshaded part and $200 in two shaded parts, which helped him know that each 1/3 equals $100. He continued the strategy to represent 3/5 for the house payment in three shaded parts and $300 in two unshaded parts, which helped him determine that each 1/5 is $150. In the last step, however, Tony erroneously thought that 1/6 of the total is $125, when 5/6 equals to

$750. Similar errors were observed in last a few steps of another PST's strategy. These errors indicate that it is challenging to keep track of the whole numbers represented as different fractional parts of multiple wholes in a multistep problem.

We observed both the second and third types of difficulties (i.e., understanding changing wholes and coordinating dif- ferent sized parts) in two other invalid strategies. For example, Rachel pictorially represented the amount for each expense using a multiple whole model but did not draw the two sixths in the third row proportionally (Fig. 14). When she was con- necting the whole number dollar amounts to the fractional parts starting from the bottom row to the top, Rachel correctly figured out that the rectangle in the fourth row represents $200, and the three strips in the third row represent $300 because each strip in the third row represents $100. She then erroneously determined that the 5/6 in the second row represented $600 because there are three more strips in the second row, and she thought each row strip represents $100. This led her to conclude that the total paycheck would be $700. Her error appears to be related to first two types of challenges because it is related to coordinating 3 equal parts out of 2 equal parts as well as connecting the whole number $300 to the pictorial representations, where $300 is represented as three equal parts in the third row and two equal parts in the second row, and they do not look the same size in the picture.

4. Implications and conclusion

4.1. Implications for research

The results showed that the PSTs in this study constructed a wide range of pictorial strategies utilizing different models for fractions, approaches to the problem, and representations of their multiple steps. In addition, they were relatively more successful in identifying an unknown referent whole using pictorial representations compared to PSTs in the previous studies by Luo et al. (2011) and Tobias (2013). We conjecture that it may be related to the nature and structure of the task in this study as well as the environment that the task was given in.

First, the task in this study was for PSTs to construct their own pictorial representations for a word problem. The tasks in the studies by Luo et al. and Tobias were to interpret given pictorial representations. Although the tasks in all three studies were about pictorial representations of wholes of fractions and fractional operations, they were clearly different in nature. The task of generating their own pictorial representations may have supported PSTs in this study as they have been shown to support elementary students for fractional concepts (e.g., Empson & Levi, 2011; Empson, 1995). In addition, the mathematical structure of the paycheck problem may have also aided PSTs in constructing valid pictorial representations. Similar to Mack's (2001) work with fifth graders, two of the three steps in the paycheck problem involved multiplying a/b × b/c, which requires the PSTs to conceptualize fractional amounts as embedded within a composite unit without the need to partition. In their drawings, PSTs assigned fractional amounts to remaining portions of their paycheck by shading or redrawing the remaining portions.

Moreover, the context of the paycheck problem might have helped the PSTs to construct a valid pictorial strategy and allowed them reason about the relationship between the fractional quantity representing each expense and the referent whole for the corresponding expense. Multiple sources have documented that word problems help support young students' thinking and reasoning about number and operations as well as fractions (e.g. Carpenter, Fennema, Franke, Levi, & Empson, 2015; Huinker, 1998). Perhaps reasoning about fractions through a familiar context, a paycheck, helped PSTs make sense of the task. In addition, the nature of the paycheck problem is different than other documented pictorial tasks. Luo et al.'s (2011) task involved a fraction multiplied by fraction with the result unknown whereas the paycheck problem involved a known result and known fractional multiplier with an unknown starting value. Lastly, the PSTs in this study worked on the task in a class environment task in the middle of a semester after several weeks of problem solving focused instruction, whereas the PSTs in the study by Luo et al. were in an assessment environment.

We believe that the key contribution of the findings of this study is the wide range of pictorial strategies that PSTs constructed when they were tasked to generate a pictorial representation for a multiplicative word problem. The finding that many PSTs can and do construct meaningful strategies calls for further research on three particular topics. One is if and how these two skills of constructing pictorial representations and interpreting given pictorial strategies are related in PSTs' development of conceptual understanding of fractions and operations. It appears that PSTs who construct pictorial presentations for word problems might be able to use the knowledge in interpreting given pictorial representations, which is a necessary skill in helping their future students understand and interpret fractions.

Second, it is important to further investigate if PSTs can extend their understanding of referent wholes for fractions, as exhibited through drawings, to develop more abstract strategies, such as algebraic solutions or numeral problems for fraction multiplication and division. It is important to learn if one of the pictorial strategies identified in this study is more productive than another in supporting abstract and formal understanding of fractions and operations.

Lastly, the finding of three common misconceptions related to PSTs' invalid strategies indicates a need for more research on PSTs' misconceptions on fundamental concepts of fractions. It was surprising to learn that several PSTs thought that fractions could be added or subtracted without considering the size of each referent whole for a given fraction. It appears that PSTs' understanding of referent wholes is intricately woven with understanding of addition and subtraction as well as multiplication of fractions. Moreover, PSTs' struggle in determining 1/3 of two equal parts (see Fig. 11 for an example) confirms the struggle identified in the study of fifth graders by Mack (2001). She identified different types of tasks in multiplying fractions and discussed that if the numerator in the multiplicand is not the same as the denominator of the multiplier ($a/b \times c/d$ where $b \neq c$, i.e. $1/3 \times 2/5$ for other bills), it is more difficult than the denominator of the multiplier being identical to the numerator of the multiplicand ($a/b \times b/c$, i.e., $3/5 \times 5/6$ for house payment). Thus, more research is needed to further investigate what types of contexts and/or fraction structures will support PSTs' development of fraction multiplication specifically within situations where numerical relationships between the denominator of the multiplier and the numerator of the multiplicand are not readily perceivable.

4.2. Implications for teaching

With regard to teaching, these results highlight that PSTs utilize a variety of pictorial representations when given a multistep problem in context. Various types of pictorial representations can act as productive means for PSTs to investigate possible strategies, examine limitations of certain strategies, and develop flexible problem-solving skills. For example, the use of an area model highlights how the problem could be worked both forwards and backwards. In contrast, the multiple wholes strategy provides support in visualizing how the solution can be built from the remaining $150 back to the total paycheck. PSTs might compare working forwards to backwards and notice that it is difficult to use the working backward strategy if you want to use singular whole instead of multiple wholes.

In addition, different types of strategies can help PSTs deepen their conceptual understanding of fractions and related concepts. For example, PSTs might compare an area model to a set model and consider how set models could be devised to appropriately represent a given problem. For example, when Tami started with 6 circles to represent the unknown paycheck and was left with 2 circles to divide by 3, she increased the total number of circles to 18, which left her with 6 circles, which is divisible by 3 (see Fig. 6). This could lead to discussions on whole number division, factors, and multiples. The variety of pictorial strategies demonstrates different, yet equally valid ways of thinking and reasoning.

Lastly, PSTs' invalid strategies can encourage them to investigate common misconceptions related to wholes for frac- tions in depth. For example, PSTs can explore why the sum of fractions for the expenses was larger than one and why the referent whole for the first and second fractions are different sizes. This type of investigation may help PSTs reason through fundamental concepts of referent wholes of fractions, instead of focusing on computations or incorrect answers.

4.3. Conclusion

Our results indicate that many PSTs in this study constructed meaningful pictorial representations for a multistep word problem involving fraction multiplication with changing wholes. Their drawings demonstrated a wide range of strategies in terms of how they represented changing wholes, the visual models they selected, and the order in which they worked on the problem. The classification of their valid pictorial representations reveals that most PSTs constructed their pictorial representations in the order that the problem was stated, using one polygon based on an area model.

The analyses of the invalid strategies suggest that they were related to three common misconceptions or difficulties—making sense of changing referent wholes, coordinating different sized parts, and connecting pictorial representation with whole number quantities. The first observed challenge in this study concurs with the documented difficulty in defining wholes for each fraction and materialized in PSTs' incorrect pictures as well as their attempts to add or subtract the given fractions (e.g., Luo et al., 2011). The second observed challenge concurs with the documented difficulty in determining a/b of c/d where $b \neq c$ in Mack's study (Mack, 2001) with fifth graders.

In comparison to previous studies, the relative success of the PSTs in this study indicates the potential benefits of PSTs generating their own pictorial representations for contextualized problems before they interpret teacher-given pictorial representations. We believe that PSTs' experiences of generating their own drawings may not only help them deepen their understanding of fractions and operations, but may also help them be more positive about providing their future students with similar learning opportunities. We are encouraged to see how the findings of this study provide potential implications for future research as well as teaching fractions to PSTs in our current classes.

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