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Conceptual Models for Integer Addition and Subtraction

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Introduction

Teachers often use contexts for helping students think about and make sense of integers and their operations (e.g., Battista, 1983; Janvier, 1985; Liebeck, 1990; Linchevski & Williams, 1999; Schwartz, Kohn, & Resnick, 1993; Tillema, 2012; Whitacre et al., 2011). Upon examination of Grade 5 and 6 California-adopted textbooks, Whitacre et al. (2011) found that nearly 90% of the textbooks used money, elevation, and temperature as contexts for instruction. Even the authors of the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) suggest credit or debit, temperature, elevation, as well as electron charges, for use with integers. But are students able to make sense of integers with these traditionally advocated contexts (e.g., money, elevation, temperature)?

Like others, we questioned if the recommended use of contexts with integer instruction is relevant or meaningful for students. Despite the prevalent use of contexts with integer operations, contexts for integers have often been criticized for being contrived (Ball, 1993). This may be related to limitations in physical embodiments of the integers (Martínez, 2006; Peled & Carraher, 2008; Vig, Murray, & Star, 2014). Even when contexts are not contrived, students may not naturally use integers when solving problems with typical contexts for integer addition and subtraction—like borrowing or owing money (Whitacre et al., 2015). Noticing the complicated nature of integers and contexts, we turned our attention to the literature on student thinking about integers. This literature (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016; Bofferding, 2014) has demonstrated that students are capable of robust reasoning about integers, but is mostly situated in how students make sense of integers in symbolic settings (e.g., how they solve problems like -2 – -5 = ☐) or pre-determined contexts.

A Survey of the Literature on Integers and Contexts

Challenges of Thinking about Integers within Contexts

One challenge is that although certain contexts or situations may evoke the use of negative integers with adults, students may not perceive the need for the use of negative integers (Mukhopadyhyay, Resnick, & Schauble, 1990; Peled & Carraher, 2008; Whitacre et al., 2015). Whitacre et al. (2015), for example, found that many students solved a debit and credit problem, in which negative integers are typically used, with positive integers only. In that study, middle and high school students correctly solved the debit and credit problem with positive integers, but they were unable to connect the negative integers to the contexts. Peled and Carraher (2008) reported a similar situation in which prospective teachers used only natural numbers for correctly solving a problem about the difference in elevation between two cities—one below sea level and one above sea level—and did not apply negative integers.

Another challenge with understanding the role of contexts is that students often draw on unconventional contexts (Mukhodpadhyay, 1997)—contexts not typically in curriculum or standards. Students are capable of using integers productively in unconventional contexts (Whitacre et al., 2012). The students in Mukhopadhyay's (1997) study invented stories without opposites that did not support integer addition and subtraction, such as representing socks as negative numbers and shoes as positive numbers. Yet, Whitacre et al. (2012) demonstrated that young students productively made sense of integers in the unconventional context of happy and sad thoughts, where the students did use opposites.

Yet, even if students understand connections between the contexts and integer operations, their understanding may not transfer to other mathematical contexts. For example, Mukhopadyhyay, Resnick, and Schauble (1990) presented students with a story related to a person owing money. The majority of students in the study understood the context and could accurately compute the person's debt. However, when presented with parallel equations out of that context, students were less successful in correctly solving the problems. A related challenge is that the ways that students use the integers may not be directly related to the contexts. For instance, when Chui (2001) presented students with a stock market task, students used metaphors of motion to represent opposing objects for these monetary transactions. This way of solving is not directly parallel to the situation given.

Additionally, students find posing stories for integer addition and subtraction number sentences to be challenging (Kilhamn, 2008; Mukhopadhyay, 1997; Rowell & Norwood, 1999). Mukhopadhyay (1997) studied children's use of storytelling as a sense-making activity of negative numbers. She asked 32 students in Grades 5 through 8 to solve expressions in sets (e.g., -3 + -4, -3 – 4, -3 – +4) involving negative integers and then to tell stories that matched the expressions. Most students in the study found the task of creating stories for equations troublesome. They often created their stories with uncertainty and recognized that their stories did not make sense or match the given equation. Some students changed an equation to an equivalent expression (e.g., -3 – -4 to -3 + - 4) before they could create a story. Although students in the study could successfully solve integer equations (i.e., procedurally obtain correct solutions), the results of Mukhopadhyay's study highlight that creation of stories for the equations proved to be difficult—potentially highlighting a gap in their conceptual understandings. Peled and Carraher (2008) reported on prospective teachers' abilities to write a story problem for 2 – 7. Prospective teachers mostly used money as a context, followed by height and temperature. Although four of 15 prospective teachers maintained the structure of the expression 2 – 7, seven of the prospective teachers wrote stories that coincided with the expression 7 – 2 and the rest did not answer. Similarly, Kilhamn (2008) asked 99 prospective teachers to pose stories for integer number sentences. Kilhamn found that only a small number of the prospective teachers used a model or context to explain the mathematics rather than posing a story—indicating the challenges of connecting integers to contexts. Those who used a model or context did so with either number lines or temperature to explain their solution. Kilhamn speculated that number lines and temperatures represent models and contexts, respectively, that are more intuitive for students.

We know that both children and prospective teachers find posing stories for integer addition and subtraction challenging (e.g., Kilhamn, 2008). They may pose unrealistic stories (Mukhopadhyay, 1997) and change the structure of the number sentences (Rowell & Norwood, 1999). When working with integer operations in contexts, students may not think about the integers as adults do (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop et al., 2016; Bofferding, 2014). Students may be unable to transfer integer understanding in one context to another context (Mukhopadyhyay et al., 1990). They may use metaphors that are not parallel with a given equation and may need to think about integers or integer equations in different ways in order to work within a particular context (Chui, 2001). We know that students struggle with relating integers to contexts (Mukhopadyhyay, 1997) and often do not use negative integers when given a conventional context, like debt (Whitacre et al., 2015).

Problem Posing: Uncovering Conceptual Structures about Integers

Although understanding integer operations is complex (e.g., Piaget, 1948) and investigating thinking about integers within contexts is even more complex (e.g., Whitacre et al., 2015), we need to look more intensely into the ways that students pose stories for integers and the conceptual structures or processes behind the problem posing (Bell, 1984; Cai, Hwang, Jiang, & Silber, 2015). Important prior work has identified ways that students think about integers (Bofferding, 2014), especially with symbolic-only problems (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop et al., 2016). However, we need to look more closely at the ways students think about and use integers *as they apply contexts to them*, especially as they may not use integers when given ready-made contexts (Whitacre et al., 2015). Identifying the ways that students think about and use integers as they apply contexts to them provides insight into students' conceptual structures (Bell, 1984) and cognitive processes (Cai et al., 2015), rather than just their strategies or procedures. A conceptual structure in this study is interpreted as the underlying mathematical thinking behind posing stories (Bell, 1984; Cai et al., 2015).

Posing stories about integer addition and subtraction is a way to make sense of student thinking about integers and contexts (e.g., Mukhopadhyay, 1997; Roswell & Norwood, 1999), a form of problem posing (e.g., English, 1997, 1998), and the stories posed are influenced by these conceptual structures (Cai et al., 2015). Because posing stories is cognitively demanding, it provides perspective into the ways that student think about and use integers—providing insight into students' thinking (e.g., Cai et al., 2015; English, 1997, 1998). Posing stories for integer addition and subtraction number sentences is a semi-structured type of problem posing (Stoyanova & Ellerton, 1996). The present study reported here extends the Mukhopadhyay

(1997) study by characterizing the nuances about the ways that students pose stories for integer number sentences and the broad ways that the students thought about and used the integers as they posed stories—a need in the field not only for thinking about integer addition and subtraction, but also for problem posing (Cai et al., 2015). Roswell and Norwood (1999) reported that prospective teachers changed the number sentence itself before posing the story and Mukhopadhayay (1997) showed that students did not think their stories matched their number sentences. These results indicate that part of describing the thinking that the students used as they posed stories requires more in depth descriptions of the ways that students posed the stories—as a way to gain "fine-grained knowledge of how they go about posing those mathematical problems" (Cai et al., 2015, p. 14). In this study, we examined how students connected integers to various contexts in order to extend the literature on students' thinking about integers. Because the aim of this present study is describing the thinking of the students as they posed stories and how they posed these stories, this points to the need to build models of thinking about integer addition and subtraction.

Previous frameworks for thinking about integers have each been developed by studying children or students' thinking within symbolic settings (e.g., solving problems like 2 – ☐ = -5 or 2 – -5 = ☐), either primarily or exclusively (e.g., Bishop et al., 2010, 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bofferding, 2010, 2011, 2012, 2014; Gallardo, 1994, 1995, 2002; Peled, 1991). Gallardo (2002) provided a framework for thinking about integers; Bofferding (2014) developed a set of children's mental models for order and value of the integers; and, Bishop, Lamb, Philipp, Whitacre, Schappelle, and Lewis (2014) generated a framework for the reasoning of children for integer addition and subtraction.

A Need for Building of Models of Integer Addition and Subtraction from Problem Posing

Given the robust challenges of students' thinking about integers within contexts (e.g., Whitacre et al., 2012, 2015), we still need insight into students' thinking about integers as they connect the integers to contexts. Students do not naturally employ use of negative integers when given contexts like debts and credits (Whitacre et al., 2015). Contexts used with negative integers are often criticized for being contrived (Ball, 1993), yet students are able to use negative integers with contexts that are not typically used—such as happy and sad days (Whitacre et al., 2012). The challenges are two-fold: use of contexts with integers is complicated, and current frameworks about thinking about integers are built from symbolic settings singularly. We need to explore the ways that students connect integers to contexts and create a framework of thinking generated from students' use of contexts as well, especially given the complicated nature of integers in contexts.

Theoretical Groundings for Building Models of Student Thinking

Our theoretical grounding for building models of student thinking from posed stories is influenced by the desire for insight into conceptual structures of students (Bell, 1984; Cai et al., 2015) as they pose stories for integers. These models are generated through exploratory model building (e.g., Cobb & Steffe, 1983; Thompson, 1982).

Conceptual Structures: Thinking Behind Problem Posing

When a student applies a particular context to an integer addition and subtraction number sentence, what is the conceptual structure that the student is drawing on? Bell (1984), one of the early researchers on student thinking about integers, wrote:

> It is also becoming clear (though I am not sure that this has been documented) that situations which are structurally identical when fully mathematized are by no means similar when first perceived; for example, money and temperature problems are differently perceived, although they both involve states and changes in directional quantities. It follows that we need to consider much more seriously than we have previously done, the development of conceptual structures in one context, then another, then perhaps exploring isomorphism. (p. 56)

Bell (1984) advocated for investigations into the conceptual structures as students used contexts with integer addition and subtraction. Decades later, Cai, Hwang, Jiang, and Silber (2015) similarly advocated for investigations into the conceptual structures behind problem posing. Also, Bell (1984) highlighted that although students may use different contexts some of these different contexts may support singular mathematical ways of thinking. That is, there may be an "isomorphism" behind ways of thinking about integers in different contexts.

Although we know about the challenges that students encounter when thinking about or using the integers within a given context (e.g., Whitacre et al., 2012; 2015), we know little about the reverse—how students apply contexts when given integers. We need insight into the conceptual structures behind students applying contexts to integer addition and subtraction. Because we seek, in this study, to build models of thinking (e.g., Steffe & Thompson, 2000; Ulrich et al., 2014) that highlight the conceptual structures (Bell, 1984; Cai et al, 2015), we name these models of thinking that we explored: *conceptual models of integer addition and subtraction*.

Tenets for Model Building and a Framework for Constructing Models

Thompson (1982) stated that researchers "must build models that would substitute for the mathematical student so that he (the researcher) may perceive the world of mathematics through the (modeled) student's eyes" (p. 152). These models are descriptions of "thinking about others' thinking" (Ulrich et al., 2014, p. 329). Thompson (1982) provided five questions for the guidance of building models of student thinking:

1. "Does the report specify a framework for constructing models?"
2. "Are the prototypes made clear?"
3. "Is the framework grounded in data?"
4. "Are the models viable?"
5. "Are the models sufficient?" (p. 160).

These questions guided our model building.

Semi-structured type of problem posing (Stoyanova & Ellerton, 1996) is recognized as a way for accessing the students' conceptions of integer addition and subtraction (Cai et al., 2015). Using semi-structured problem posing is the first part of identifying a framework for building models specified by Thompson (1982). Semi-structured problem posing includes providing students with an opportunity to make sense of mathematics in an open situation. Having students, for example, pose stories for open number sentences (Mukopadhyay, 1997) is this type of semi-structured problem posing. By providing the structure of the open number sentences, we ensured students' opportunities of using negative integers, which they might not have otherwise (see, e.g., Whitacre et al., 2015). Freedom in the choice of context as students pose stories provides opportunities for the students to draw upon their desired conceptual models. Furthermore, part of the framework for building conceptual models includes the assumption that conceptual structures may be uncovered as students pose integer addition and subtraction stories with a variety of contexts (Cai et al., 2015). More specifically, a variety of stories with different contexts may be produced with a similar conceptual structure (Bell, 1984). Although different contexts are perceived differently, a singular conceptual structure may be behind the variety of posed stories (Bell, 1984). The framework includes the examination of stories posed for integer open number sentences as insight to the conceptual structures behind problem posing (Cai et al., 2015). In this study, with the aim of generating conceptual model descriptions (Cobb & Steffe, 1983), conceptual models may be uncovered through examination of posed stories for integer addition and subtraction number sentences with attention to tenets of model building (Thompson, 1982).

Research Question

Examining students' posed stories led to the following research question: What conceptual models of integer addition and subtraction descriptions can be generated from students' posed stories?

Method

One way for exploring how students connect integer addition and subtraction to contexts is through a generative study. Generative studies "allow investigators to develop new categories for description" (Clement, 2000, p. 558). Because students find integrating integers with contexts challenging (e.g., Chui, 2001) and students do not necessarily use negative integers when given a context (Whitacre et al., 2015), a generative study (Clement, 2000) provides the opportunity for an exploratory study in the creation of an initial set of models (e.g., Steffe & Thompson, 2000; Thompson, 1982). A generative study offers space for descriptions about how students think about contexts in the realm of integer addition and subtraction, which may be different than how they reason in symbolic situations. Clement (2000) described, "The purpose of a generative study is to generate new observation categories and new elements of a theoretical model in the form of descriptions of mental structures or processes that explain the data" (p. 557). Thus, a generative study provides freedom for the researcher in the exploration of thinking about integer addition and subtraction in relationship to contexts, in order to develop a set of descriptions about students' thinking.

Participants

Six eighth-grade students from a rural public school in a Midwest region of the United States participated in this study. We examined how integers are applied to contexts. Because students prior to instruction may not know how to apply negative integers to contexts or would possibly create only unconventional ways to apply the integers to the contexts, we decided to interview students that had experience with integers prior to our study. Furthermore, we selected the six participants for the study because the students' mathematics teacher considered them to be advanced mathematically and thus we thought they might be more apt than their peers to provide a variety of stories for analysis. The teacher shared that the students had previous experiences operating with the integers and included contexts (i.e., debit or credits, electrons, elevation, temperature) advocated by the authors of the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010). The students' perceived mathematical status provides an interesting perspective into investigating conceptions of integers because they were considered to be mathematically proficient. Additionally, these students provided insight into students' thinking after classroom instruction about integers. We were satisfied with the small sample of students because we viewed this study as an initial study on generating models, with the expectation that there would be follow-up studies in later model refinement (Clement, 2000).

Interviews

For generating models of student thinking, Thompson (1982) emphasized that these should be "grounded in data" (p. 160). Thus, the first author employed a semi-structured clinical interview with each of the six eighth-grade students (Goldin, 2000). Students posed a story for each integer addition or subtraction open number sentence. The first two open number sentences given to students included only positive integers. These problems were given to students first to determine if they struggled with problem posing. After students demonstrated proficiency with posing stories for the open number sentences with positive integers only, students were asked to pose stories for nine to 10 open number sentences involving both positive and negative integers. Each student was interviewed once with interviews lasting approximately 30 minutes.

Data Analysis

Generative studies often lead to the interpretative analysis of transcripts of clinical interviews (Clement, 2000). For the analysis of research questions, Clement (2000) suggested a generative analysis for exploratory studies in model development in order to generate a set of descriptions about thinking. To do this, we first transcribed the interviews, generated themes and descriptions for the conceptual models, and continuously negotiated the themes and descriptions, developing the clarity of the models (Thompson, 1982). We then coded each of the stories with our conceptual model descriptions and wrote student profiles as a check on the sufficiency of the models (Thompson, 1982). Our generative analysis aligned with analyses recommended for second-order models (see, e.g., Thompson, 2008) because we spent significant time in the continuous development of the models and descriptions, with over five iterations of reflective development of the descriptions. We will give more insight into building model descriptions in the next section and describing how the students used conceptual models in the results.

Building Model Descriptions

For analysis of our research question, each individual story corresponding to an open number sentence constituted the unit of data. We examined each of the 56 stories, generating emergent themes about ways that students used, or possibly conceptualized, the integers in the stories. We "formulated an initial description of the subject's mental structures, goals, and processes that provides an explanation for the behavior exhibited in the transcript" (Clement, 2000, p. 575). These initial descriptions constituted themes that emerged from within each story posed (e.g., balancing integers, using debits and credits, using movements). After we looked for emerging themes individually and collectively, we discussed which themes would be used for generating descriptions of conceptual models. We then considered and discussed the meanings of zero, positive, and negative integers for each particular story and across stories. We drew upon Thompson's (1982) question about model building: "Are the prototypes made clear?" (p. 160). As we generated descriptions from themes and sought to produce clarity in model definitions, the discussions about developing descriptions included unpacking how the students made use of positive and negative integers. Some of these discussions compared how the students incorporated positive and negatives integers as gains and losses similarly in different contexts (e.g., losing a pencil is similar to losing a penny). We discussed how the students used the integers in ways that balanced each other out (see, e.g., Whitacre et al., 2012). But, we noticed this differed from their use of debits and credits, and we needed to distinguish that in the descriptions of the models. That is, the students treated comparing good and bad deeds during a week (e.g., comparing two objects) differently than losing pencils (e.g., applying a change of quantity to one object). We also discussed how the students used ideas of movement and travelling (see, e.g., Thompson & Dreyfus, 1988; Ulrich, 2012) and how those ideas also seemed different than the thinking one might use to describe losing a pencil or comparing good and bad deeds during a week. To help solidify these differences in thinking, with clarity of the models in the forefront, we distinguished the roles of the positive integers, negative integers, and zero that the students attributed in their posed stories in our descriptions.

As we worked on transforming these themes into descriptions, we noticed one of the students' stories did not fit into the initial themes we generated (i.e., balance, debits and credits, movement) because the story supposed integer use in a positional way. This positional use of integers did not make use of movement, balance, or debits and credits. This revealed that our model descriptions were not sufficient enough (Thompson, 1982). Although this was only a singular story in our data, the research literature (Gallardo, 2002) and historical use of integers (e.g., Day & Thomson, 1843) supported the mathematical thinking behind posing this type of story. From this, we created a description of a model in which the integers are used in relative positions and created a set for supporting descriptions of models that sufficiently described all of our data. As we discussed how the students used the positive integers, negative integers, and zero differently in these various posed stories, we created and refined descriptions that emphasized this.

Conceptual Models Built from Posed Stories

The following conceptual models emerged from this study: Bookkeeping, Counterbalance, Relativity, and Translation. These conceptual models for integers and integer addition and subtraction represent ways that the students used and thought about integer addition and subtraction, and ultimately negative numbers.

Bookkeeping

A conceptual model of *Bookkeeping* describes integers used with gains and losses. An example of the Bookkeeping conceptual model is the borrowing and receiving of money. Debit or credits of money are a prominent context when discussing negative integers (e.g., Whitacre et al., 2012); however, the Bookkeeping conceptual model represents a gain and loss of anything and is not necessarily limited to the context of money. For example, gains and losses can be conceptualized with the owing and gaining of candy bars or wanting and receiving baseball cards. Zero in the Bookkeeping conceptual model represents either no gain or loss. Examples of the Bookkeeping conceptual model are illustrated in these students' stories:

> Isaac: *Lewis was not really good at keeping his homework coming in. He owed two assignments already. And later on in the day, he realized he still had three more assignments due. And in total he owed five assignments.* (-2 – 3 = ☐)
>
> Wesley: *I wanted seventeen baseball cards and I got twelve. Now, I want so many more.* (-17 + 12 = ☐)

Isaac used the positive integers to represent turning in assignments and the negative integers to represent owing assignments. Wesley's use of integers as gain or losses is highlighted by the "want" of seventeen baseball cards and the "gain" of twelve cards. Bookkeeping appeared to be the most used conceptual model by the students in this study (used in 49 of 56 stories, see Table 2).

Counterbalance

A conceptual model of *Counterbalance* describes use of integers that balance or "cancel" each other out. Positive and negative integers in Counterbalance are not just opposites, but opposites that balance and neutralize each other. The zero in the Counterbalance conceptual model indicates neutralization. The distinguishing element of the Counterbalance model is that the quantities always remain present, even when neutralized. For example, consider three electrons (-3) and three protons (+3) that provide an electrical charge of 0. The corresponding mathematical equation is -3 + 3 = 0. The electrons, with a charge of -3, and the protons, with a charge of +3, still *exist* despite the neutralization. This existence of the quantities that remain, but are neutralized, differentiates this way of thinking from Bookkeeping or the other models. Students did not use the counterbalance conceptual model frequently in this study (2 out of 56 stories, see Table 2) and is illustrated in the stories below:

> Isaac: *Joe did some bad things in the past. And, he's trying to even out the scales by doing good things. So far, he still has five bad things to re-pay for what he's done. And, he can think of 26 good deeds. He does the 26 and by the end of week he's evened out the scales and did more than he expected. He did 21 more good deeds.* (-5 + □ = 21)
>
> Joseph: *You're playing football and you had 18 kids wearing white jerseys and, you'd have to have 13 kids wearing black jerseys. The white team has 18 jerseys and the black team has 13 jerseys. And, the neutral team, which would be the equals, would only have 5 kids left*. (18 + □ = 5)

In Isaac's story, the integers are being used in a way that creates balance—a good deed balances a bad deed. Although these deeds "evened out the scales," the deeds are still present. Similarly, with Joseph's story, the football players wearing black jerseys balance the football players wearing white jerseys.

Relativity

A conceptual model of *Relativity* describes the use of integers with comparative positions to an unknown referent. With Relativity, the unknown referent may be treated as zero, and the integers are descriptions of reference from that zero. Consider, for example, the temperature scale, specifically -2 degrees in both Celsius and Fahrenheit. Both are measures from the positions of zero and represent different temperatures from each other. The decision of the temperature scale is not something fixed. With Relativity, the zero is not an absence of a quantity. Rather, the zero in the Relativity conceptual model is an arbitrarily or specifically selected point of comparison. Students did not draw on Relativity frequently in this study. In fact, the example below is the only use of Relativity demonstrated in this study:

> Joseph: *I guess you could do baseball ... Say you are down [five] runs in the first inning ... And you end up losing by fifteen runs. So you would have to have ten runs in the other innings to be down by fifteen runs*. (-5 + □ = -15)

In this story, the score of the actual game is not known. Rather, the negative integers are used in a way relative to a tied game. This unknown score of the tied is the reference or the chosen zero. The score could have been 10 to 10, 22 to 22, or an infinite number of possibilities. The integers were used in comparative ways to this unknown referent. Although there was only a singular use of the Relativity conceptual model this study, there is historical and evidence of mathematical use of integers as relative numbers (e.g., Gallardo, 2002).

Translation

A conceptual model of *Translation* describes the use of integers when they are treated as directed numbers or vectors. With the Translation conceptual model, integers are used to shift any kind of mathematical objects (e.g., a number, a point, a curve). In contrast to the Relativity conceptual model, which does not include movement, the Translation conceptual model emerges from the contexts of traveling or moving about in a linear model, coordinate plane, or three-dimensional space. The zero in this conceptual model is a zero vector, or no movement. Similar to Relativity, the zero can also represent a relative number with the positive and negative integers indicating a translation in one direction or another from the relative zero. Students used the Translation conceptual model in 4 of the 56 stories. All four examples using a Translation conceptual model in this study are illustrated below:

> Joseph: *Let's say, you are going to your family's house for Christmas and you're travelling down the road… the numbers would be the miles and you accidentally turned in the wrong direction. And so the further and further away would be the larger the negative number. So, first you take a right and go negative fourteen miles away. And then, you take another right and go negative seven miles away. So total you are negative twenty-one miles away.* ($-14 + -7 = \square$)
>
> Drake: *Ok, a car, it is a racecar, is behind the finish line two feet. It goes back another three feet and is now negative five feet behind the finish line.* ($-2 – 3 = \square$)
>
> Drake: *An airplane is eight miles above sea level and then goes twenty miles down. And is now negative twelve miles below sea level.* ($8 – 20 = \square$)
>
> Drake: *So the shark is negative fourteen feet below sea level and goes up twenty feet. He is then jumping out of the water, six feet about the water.* ($\square – -20 = 6$)

In Joseph's story the integers represent linear movement in the right direction (towards the family's house) or the wrong direction (away from the family's house). Similarly, Drake's story incorporates linear movement with a sharks and airplanes travelling vertically, moving up and down, and a racecar moving horizontally backwards and forwards. Distinguished from Relativity, the referent is known rather than unknown. That is, in both Joseph's and Drake's stories, the referent is established. The referent or zero for Drake's shark story, for example, is sea level. Albeit this is still a relative use of the integers, this is a more positional use related to making sense of movement, in contrast to Joseph's baseball story that used relativity without movement, and we described this as using a Relativity conceptual model.

Summary of the Conceptual Models for Integer Addition and Subtraction

The conceptual models for integer addition and subtraction represent ways that these students used the integers. These ways of thinking about and using the integers are summarized in Table 1.

Table 1

The Conceptual Models for Integer Addition and Subtraction

Conceptual Model	Description of Model
Bookkeeping	A conceptual model of *Bookkeeping* describes using integers with gains and losses. Zero in the Bookkeeping conceptual model represents either no gain or loss.
Counterbalance	A conceptual model of *Counterbalance* describes use of integers that balance or neutralize each other out. Positive and negative numbers in Counterbalance are not just opposites, but opposites that balance or neutralize. The distinguishing element of the Counterbalance model is that the quantities always remain present, even when neutralized. This existence of the quantities that remain, but are neutralized, differentiates this way of thinking from Bookkeeping or the other models. Zero in the Counterbalance conceptual model indicates neutralization.
Relativity	A conceptual model of *Relativity* describes the use of integers with comparative positions to an unknown referent. With Relativity, the unknown referent may be treated as zero, and the integers are descriptions of reference form that zero. Consider, for example, the temperature scale. With Relativity, the zero is

	not an absence of a quantity. Rather, the zero in the Relativity conceptual model is an arbitrarily or specifically selected point of comparison.
Translation	A conceptual model of *Translation* describes the use of integers when they are treated as directed numbers or vectors. With the Translation conceptual model, integers are used to shift any kind of mathematical objects (e.g., a number, a point, a curve). In contrast to the Relativity conceptual model, which does not include movement, the Translation conceptual model often emerges from the contexts of traveling or moving about a linear model, coordinate plane, or three-dimensional space. The zero in this conceptual model is a zero vector, or no movement. Similar to Relativity, the zero can also represent a relative number with the positive and negative numbers indicating a translation in one direction or another from the relative zero.

How Students Posed Stories and the Relationship to Conceptual Models

In the following section, we provide two profiles of the six students that participated in this study. We highlight one student who posed stories that supported one conceptual model and highlight another student who posed stories that supported a variety of conceptual models. Table 2 highlights the conceptual models that all of the students in this study used. Bookkeeping was used more than other conceptual models (see Table 2) and is difficult to distinguish from Counterbalance, which is why Isaac and the Bookkeeping Conceptual Model (and his singular use of Counterbalance) is the first profile. Similarly, half of the students used more than one conceptual model, which is why Joseph and how he used different conceptual models is highlighted in the second profile. The various uses of contexts, changes of number sentence structure, and realism are described for each student to better highlight the nature of the conceptual models.

Table 2

Conceptual Models for Integer Addition and Subtraction used by the students for each number sentence (B = Bookkeeping; C = Counterbalance; T = Translation; R = Relativity)

	18 + □ = 5	-17 + 12 = □	-14 + -7 = □	-5 + □ = -15	8 – 20 = □	5 – □ = 17	-2 – 3 = □	□ – -20 = 6	-10 – -22 = □	-5+ □ = 21
Joseph	C	B	T	R	B	B	B	B	B	Not Asked
Hailey	B	B	B	B	B	B	B	B	B	Not Asked
Drake	B	B	B	B	T	B	T	T	B	Not Asked
Dana	B	B	B	B	B	B	B	B	B	B
Isaac	B	B	B	B	B	B	B	B	B	C
Wesley	B	B	B	B	B	No Story Provided	B	B	B	B

Isaac and the Bookkeeping Conceptual Model

In Figure 1 we present the stories Isaac posed per number sentence and conceptual model. Isaac posed nine stories using the Bookkeeping conceptual model and one story using the Counterbalance conceptual model (see Isaac Story 3 [IS3]). Five of the nine stories determined to draw upon Bookkeeping pertained to money, a conventional context with integers. The other four contexts for Bookkeeping were not money, indicating a flexible use of integers with gains and losses beyond money. The other contexts he used included: worms for fishing (see IS1), good and bad deeds (see IS3), games (see IS5), hair loss (see IS6), and assignments (see IS8).

His use of deeds (see IS3) was different from some of the other contexts. In this story, Isaac used the integers to represent good and bad deeds, in which the good and bad deeds balanced each other out. Because Isaac used the integers in a way that balanced each other out, he drew on a Counterbalance conceptual model as he posed this story. This use of good and bad deeds is not a typical context used in curricular materials or standards.

Equation	Story	Conceptual Model
18 + □ = 5	**[IS1]** Benny decided to fish. He had 18 worms in his can to start out with for his fishing trip. [A]s Benny was riding his bike to the river he lost an amount of worms. When he got to the river he only had five worms. When he wanted to know how many worms he lost, he took 18 minus 5 and got 13. So, Benny finished his fishing trip sadly without the 18 worms.	Bookkeeping
-17 + 12 = □	**[IS2]** Carrie … owed seventeen dollars to the bank. And, she got a certain amount of money, twelve dollars. She repaid the bank, but still the bank told her she owed five more dollars.	Bookkeeping
-5 + □ = 21	**[IS3]** Joe did some bad things in the past. And, he's trying to even out the scales by doing good things. So far, he still has five bad things to repay for what he's done. And, he can think of 26 good deeds. He does the 26 and by the end of week he's evened out the scales and did more than he expected. He did 21 more good deeds.	Counterbalance
-14 + -7 = □	**[IS4]** Eddie went to the bank and [borrowed] fourteen dollars. Later on in the week he [borrowed] seven more dollars... When the bill came into the mail, the bank said he owed twenty-one dollars to them.	Bookkeeping
-5 + □ = -15	**[IS5]** I broke five games. And then the next day I broke some more games, ten of them to be exact. It turns out the owner, that happened to be there, found out I broke fifteen games. That I owed him.	Bookkeeping
8 – 20 = □	**[IS6]** Sonic the Hedgehog had 8 hairs. When he grew up, he found out that if he had more hairs he would have lost	Bookkeeping

	twenty more. And, if he, if that did happen, he would have lost twelve hairs.	
5 – ☐ = 17	**[IS7]** Jerry figured out that if he lost five dollars in this bet, that the five dollars that he had, which … he would be down to nothing. So, he went all in. He did lose that [hand], but he wanted to continue. So he took a loan for twelve dollars … He took a loan for twenty-one dollars. I mean, twenty- two dollars.	Bookkeeping
-2 – 3 = ☐	**[IS8]** Lewis wasn't really good at keeping his homework coming in. He owed two assignments already. And later on in the day, he realized he still had three more assignment due. And in total he owed five assignments.	Bookkeeping
☐ – -20 = 6	**[IS9]** Jay noticed that … he had fourteen dollars. And he owed a friend twenty. He paid the friend back fourteen dollars and still owed six.	Bookkeeping
-10 – -22 = ☐	**[IS10]** There was a man and he owed ten dollars to the mob. Um, he paid twenty-two dollars, which he knew was over. But, he thought he would have some extra points if he paid back more. The mob gave back his twelve dollars. And, he had an extra twelve dollars to use.	Bookkeeping

Figure 1. Isaac's stories by number sentence with conceptual model use.

At times when Isaac posed stories, he changed the structure of the number sentences (i.e., IS1, IS6, IS7, IS8, IS9, IS10). For example, Isaac changed -2 – 3 = ☐ to -2 + -3 = ☐ and posed a story about late homework assignments in which he started with two late assignments and gained three more late assignments and represented late homework with negative integers (see IS8). However, the number sentence -2 + -3 = ☐ represents Isaac's story better than -2 – 3 = ☐. Similarly, the number sentence 5 – ☐ = 17 (see IS7) does not seem to fit his story in which a man loses five dollars and takes another loan. Yet, he still drew upon the Bookkeeping conceptual model because he used the integers as gains and losses.

Although he used the Bookkeeping conceptual model most frequently in his stories, he did not always use it appropriately (i.e., IS6, IS10). Of the 10 stories Isaac produced, a few of the stories used positive and negative numbers in unrealistic ways. For example, for 8 – 20 = □, he discussed how Sonic the Hedgehog lost hairs that did not exist (see IS6). Because Sonic cannot lose hair he does not have, this story is not realistic. Even when Isaac used contexts like money, which is considered a more conventional context than hair loss for integer operations, the stories with money were not entirely realistic. For example, for the number sentence -10 – -22 = □, Isaac posed a story about borrowing 10 dollars from the mob and repaying more (IS10). Although it is possible to borrow a small amount of money from the mob, this would likely not happen. Isaac changed the number sentence structure (IS1, IS6, IS7, IS8, IS9, IS10) and posed unrealistic stories for two of them (IS6, IS10). Although Isaac only used the Counterbalance conceptual model once as he posed his stories, he was the only participant that drew only on Counterbalance and Bookkeeping.

Joseph and the Bookkeeping, Counterbalance, Translation, and Relativity Conceptual Models

Joseph mostly relied on Bookkeeping conceptual model use in six of the nine stories he posed, but he also made use of various other conceptual models—Counterbalance, Translation, and Relativity (see Figure 2). Of his uses of Bookkeeping within the stories, Joseph used contexts that focused on the increases or decrease of amounts, such as gaining stains and removing stains on pants (see Joseph story 7 [JS7]) or having hunger and satisfying hunger (see JS2). However, Joseph did pose one story about borrowing money (see JS5).

Equation	Story	Conceptual Model
18 + □ = 5	**[JS1]** So you're playing football and you had 18 kids wearing white jerseys and… you'd have to have 13 kids wearing black jerseys. So the white team has 18 jerseys and the black team	Counterbalance

	has 13 jerseys. And the neutral team which would be the equals would, uh, only have 5 kids left.	
-17 + 12 = □	**[JS2]** There are 17 people in a house. And, uh, they're hungry so they want cookies. So you only make 12 cookies. And that only feeds… 12 people, of course. And then there are 5 people left that are hungry.	Bookkeeping
-5 + □ = 21	**Equation Not Given**	NA
-14 + -7 = □	**[JS3]** Let's say, you are going to your family's house for Christmas and you're travelling down the road… uh, the numbers would be the miles and you accidentally turned in the wrong direction. And so the further and further away would be the larger the negative number. So, first you take a right and go negative fourteen miles away. And then, you take another right and go negative seven miles away. So total you are negative twenty-one miles away.	Translation
-5 + □ = -15	**[JS4]** I guess you could do baseball … Say you are down [five] runs in the first inning And you end up losing by fifteen runs. So you would have to have ten runs in the other innings to be down by fifteen runs.	Relativity
8 – 20 = □	**[JS5]** I'd say that I'm going to the mall. And, there is a 20 dollar gift. And, uh, I ask my parents … I only have 8 dollars. So I'm 12 dollars out ... So, I'd have to ask for another twelve dollars from my parents to get the Christmas present.	Bookkeeping
5 – □ = 17	**[JS6]** If you have five presents under the tree ... And, you would have to have twelve additional presents to get seventeen presents under the tree.	Bookkeeping
-2 – 3 = □	**[JS7]** Say you got a new pair of pants and you got two stains on them. My mom wouldn't be very happy. And, uh, you're playing football. …So you would get another three stains on your pants which would be … a negative five, which would be a total of five stains on your pants.	Bookkeeping
□ – -20 = 6	**[JS8]** I got chickens at my house and I think of eggs, I guess. You need to find the amount and you have a total of six eggs that I brought up to the house. ... So the positive twenty would be the eggs that you gather. But, unfortunately, my dog got a little hyper as I was carrying the eggs up to the house and he made me spill negative fourteen. So, negative fourteen would be the eggs that cracked. And, positive six would be the eggs that actually made it to the house.	Bookkeeping

-10 – -22 = □	**[JS9]** Say I am going fishing. And the positive twenty-two would represent the first caught total. So I caught twenty-two fish, and then the negative ten would be the ten small fish that we weren't going to eat. I put them back in the lake. And the box would equal the fish that we were going to eat, which would be twelve.	Bookkeeping

Figure 2. Joseph's stories by number sentence with conceptual model use.

In addition to Joseph writing stories with unconventional contexts alongside various conceptual models, he often changed the number sentences to accommodate his conceptual model (see stories JS5, JS6, JS7, JS8, and JS9). For example, for the equation 8 – 20 = □, in which he posed his only story about money, Joseph implicitly changed the number sentence to 20 – 8 = □, rather then 8 – 20 (see JS5). He changed the structure of the number sentences in most of his stories, with the exception of three stories (JS2, JS3, JS4).

Joseph was one of the few participants who employed a variety of different conceptual models. He used Counterbalance (see JS1), Translation (see JS3), and Relativity (see JS4) in addition to Bookkeeping. He also used a variety of contexts. He used a context of movement paired with Translation when he described moving in the "right" or "wrong" directions (see JS3). He also used different colored football jerseys to the integers (see JS1) and negative integers to a baseball game score, which are not typically used in baseball (see JS4).

His stories were mostly realistic. Consider story JS7, in which negative integers represent stains on pants and positive integer represent cleaning stains. Although this may be considered an unconventional use of integers, it is reasonable to use integers this way. One of the Joseph's stories was unrealistic (see JS1). He used positive integers to represent "kids wearing white jerseys" and negative integers to represent "kids wearing black jerseys." Although this is a context that may have worked, he mentioned a "neutral team which would be the equals" that added an element of unreasonableness to the story. Even though Joseph often changed the structure of the number sentence, Joseph was overall quite flexible with his use of integers in conceptual model uses by applying different ones to a range of realistic contexts.

Summary of Different Conceptual Models behind the Stories Posed

The students used these conceptual models differently. Students used multiple conceptual models. Yet, some students used only one or primarily one conceptual model. Table 3 highlights how many students used the different conceptual models at least once.

Table 3

Different Conceptual Model Use by the Students

Conceptual Model	Number of students in this study that used the conceptual model at least once
Bookkeeping	6 of the 6 students
Counterbalance	2 of the 6 students
Relativity	1 of the 6 students
Translation	2 of the 6 students

As illustrated in Table 3, most students drew on the Bookkeeping conceptual model. Fewer students demonstrated use of Counterbalance, Relativity, and Translation. Although the students used these conceptual models differently, as illustrated in the student profiles, each of the students posed a variety of contexts, often changed the structure of the number sentence, and posed realistic or unrealistic stories in order to accommodate their conceptual model use.

Discussion

We asked students to pose stories for open number sentences involving positive and negative integers investigating how students thought about integers in relationship to contexts. In particular, we presented a set of conceptual models for integer addition and subtraction as a way of describing thinking about and using integer addition and subtraction (Clement, 2000; Steffe & Thompson, 2000; Ulrich et al., 2014). We then provided two students' profiles. As part of those profiles, we described the types of context, such as whether the students used recommended curricular contexts like credit or debit, temperature, elevation or electron charge or some other context. We also described how these students changed the structure of the number sentence and the realism of the stories to accommodate their conceptual model use. Although some of the students posed stories with unexpected or unrealistic contexts and changed the structure of the number sentences (Kilhamn, 2008; Mukhopadhyay, 1997; Rowell & Norwood, 1999), the students were able to make use of a variety of different conceptual models of integer addition and subtraction.

Unconventional Contexts of Posed Stories

Most of the stories involved unconventional contexts (e.g., lost homework, wanting baseball cards) or contexts not typically advocated by curricula (Whitacre et al., 2011) or standard documents (NGA & CCSSO, 2010) for integer addition and subtraction. Yet, students posed unconventional contexts in realistic ways that supported mathematical thinking about the integer addition and subtraction. The students in this study used a variety of productive and realistic contexts that were also unconventional, which may indicate that students are much more flexible in thinking about integers beyond the suggested contexts within instruction, curriculum, or standard documents. The students may produce these types of unconventional contexts to accommodate their conceptual model preference as opposed to using a more conventional context. Although it could be conjectured that unconventional stories are posed prior to formal school instruction only; the results of this study revealed uses of unconventional contexts after formal school experiences.

Contexts of Posed Stories as Insights into Thinking about Integers

Although the students used a variety of contexts, behind their posed stories are broad mathematical ways of thinking about integers (Bell, 1984; Greca & Moreira, 2000). We analyzed the stories that the students posed, with particular emphasis on the ways that the students applied and used the contexts to the integer addition and subtraction number sentence, for insight into their mathematical thinking about integer addition and subtraction (Cai et al., 2015). The contexts the students used, although not always realistic or conventional, not only provided perspective into their thinking, but also illustrated that there were common ways that the students thought about and used the integers. Many contexts in the stories posed by students related to the idea of a loss of discrete quantities. Contexts like “lost pencils,” “want of baseball cards,” and “cracked eggs” are all related in that the students posed these stories with contexts that support the conceptualization of a loss of some sort of discrete quantity. This was the case for each of the conceptual models. Students used Translation, for example, as they described sharks jumping up and down, racecars moving forward and backwards, and trips home in right and wrong directions. Although these are different contexts, those contexts evoke mathematical conceptions of linear movement.

The descriptions of the conceptual models capture the essence of Bell’s (1984) isomorphic claim by recognizing that similar thinking is behind the contextual use of “lost books” and “needing pencils.” It’s possible that students may also compromise ideas of realism of the context or even change the structure of the number sentence to better accommodate their conceptual models as well.

Significance of the Conceptual Models for Integer Addition and Subtraction

The conceptual models illustrate the types of thinking beyond operations that students may need. Other significant features of the conceptual models are described next:

1. The conceptual models describe integer thinking in a different way from other models—built from the stories that students posed.
2. The descriptions of the conceptual models have explicitly defined roles for the integers—the stories that students posed used integers in different ways.
3. There is nuanced thinking about integers that students need and use, which is beyond operation with integer alone—the ways that students, for example, posed stories that drew on Bookkeeping and Counterbalance conceptuals differently in how quantities were used.

Model building from problem posing. The development of the conceptual model descriptions stemmed from students' problem posing, which is a distinction of the conceptual models for integer addition and subtraction from other frameworks of thinking about integers. The role of contexts in relationship to integer addition and subtraction is complicated (e.g., Stephan & Akuyz, 2012; Whitacre et al., 2012). We know that children do not always use integers when given particular contexts (Mukhopadyhyay et al., 1990; Peled & Carraher, 2008; Whitacre et al., 2015). This present study extends this scholarly discussion with the descriptions of conceptual models, second-order models of the thinking behind applying contexts to integer addition and subtraction. Furthermore, this study extends the scholarly discussion on thinking about integers and contexts with descriptions into how students applied contexts when they were given integer open number sentences. Because contexts are rooted externally in our world, and the students created the contexts in this study as they posed stories (Cai et al., 2015), this framework of second-order models (Steffe & Thompson, 2000; Ulrich et al., 2014) provides insight into thinking about integer addition and subtraction that connects internal representations of their thinking (i.e., conceptual models) directly to external representations of the world (i.e., contexts).

Although research with integers spans decades (e.g., Bell, 1984; Janvier, 1985), explicit scholarly discussion on thinking about integer addition and subtraction has gained momentum recently (e.g., Bishop et al., 2016; Bofferding, 2014; Stephan & Akuyz, 2012; Wessman-Enzinger, 2018) with increased interest in research in student thinking about integers (Lamb et al., 2013), calls for different studies and perspectives on student thinking (Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled, 2014; Lamb et al., 2013), and engaging in scholarly discourse across these different perspectives (Bofferding & Wessman-Enzinger, 2015). The descriptions of conceptual models provides a distinct perspective by describing thinking about integer and addition and subtraction as students connect symbolic use of integers (open number sentences) to contexts (posed stories).

Models with explicitly defined roles of integers. The explicit definitions of the integers (i.e., positives, negatives, zero) within each conceptual model represent a significant feature of this framework on thinking about integer addition and subtraction, providing “clarity” in the models (Thompson, 1982, p. 160). Although other frameworks exist that describe thinking about integers (Gallardo, 2002), integer ordering, value, and directed magnitude (Bofferding, 2014), and integer addition and subtraction (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014), the conceptual models described in this study present explicitly defines roles of positive integers, negative integers, and zero. By examining the students' posed stories, as they connected the integers to contexts, the set of descriptions generated captured the students' thinking about integers through second-order models (Steffe & Thompson, 2000; Ulrich et al., 2014), with explicit attention to the role of positive integers, negative integers, and zero.

Table 3

The Role of Zero as Defined in the Conceptual Models

Conceptual Model	The Role of Positive and Negative Integers	The Role of Zero
Bookkeeping	Positive and negative integers represent gains and losses of a quantity.	Zero represents either no gain or loss in quantity.
Counterbalance	Positive and negative integers are not just opposites, but opposites that balance each other out to neutralize.	Zero represents either no gain or loss in a discrete quantity, but zero also indicates neutralization.
Relativity	Positive and negative integers represent comparative positions to an unknown referent.	Zero represents an arbitrarily or specifically selected point of comparison.
Translation	Positive and negative represent linear shifts of any kind of object.	Zero may represent a relative number with the positive and negative integers representing a translation in one direction or another from the relative zero, but zero also represents a zero vector, or no movement.

As Table 3 illustrates the explicit roles of the integers—positive, negative, and zero—found in the conceptual models descriptions. The Bookkeeping and Relativity definitions highlight one role of zero and the Counterbalance and Translation definitions highlight two different roles of zero. This may point to a relationship between the conceptual models; perhaps Bookkeeping and Counterbalance are related and Translation and Relativity are related because of the similar zeros. Translation and Relativity, for instance, both have zero defined as a relative zero, but Translation also has zero defined as a movement in no direction. Similarly, the descriptions of the conceptual models also explicitly captured the roles of opposites that the students used, with the definitions of the models including the role of positive and negative integers.

Distinction of Bookkeeping from Counterbalance. The distinction of the Bookkeeping conceptual model from the Counterbalance conceptual model is an example of how this set of conceptual models extends the scholarly discussion on student thinking about integers. Although we know that children often think differently from adults (e.g., Bofferding, 2014), the prior literature describing students' thinking about integers has not explicitly distinguished applying a gain or loss to a singular quantity (i.e., Bookkeeping) from the comparison of two quantities that neutralize (i.e., Counterbalance). Because this study was rooted in data from students, an element of building second-order models, and Thompson (1982) emphasized that models need to sufficiently capture what the students do, this distinction needed to be made. Instructional models typically focus on equilibrium and number lines models (e.g., Almeida & Bruno, 2014; Vig et al., 2014), but this does not adequately describe thinking of the students' use of contexts as they posed stories.

The descriptions of the conceptual models distinguish nuances between Bookkeeping and Counterbalances. The models are also built from the stories that students pose and provide defined roles of positive integers, negative integers, and zero. Although these are the distinct affordances of the conceptual models for integer addition and subtraction, future investigations into how these conceptual models are related to the different frameworks for student thinking about integers are needed (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bofferding, 2014; Gallardo, 2002). The conceptual models provided in this study, and the descriptions of how students used them, initiates that discussion.

Implications for Mathematical Thinking and Learning

The following section will highlight the implications of identifying and describing the conceptual models for integer addition and subtraction that the students drew upon and how they did this as they posed stories. The following implications discussed next are:

1. There is a need for the re-examination of the usefulness of conventional contexts in school mathematics to best support the mathematical thinking and learning of students.
2. Awareness of the conceptual models is important, as similar thinking behind the conceptual models is present in advanced mathematics.
3. Problem posing for integer open number sentences is an insightful assessment tool for describing and understanding the thinking and learning of students.

A need for re-examination of the usefulness of conventional context. Contexts are an instructional tool that many educators use with the teaching and learning about the negative integers (e.g., Liebeck, 1990; Stephan & Akuyz, 2012). Hidden behind the contexts are ways of mathematical thinking that are more comprehensive and uses that we need to be cognizant of (Bell, 1984; Cai et al., 2015). Although the contexts used in instructional settings may promote different ways of thinking about integers, or the conceptual models for integer addition and subtraction, the students in this study did not use contexts in conventional ways.

None of the students used contexts of temperature or electron charges, which are typical contexts (Altiparmak & Özdoğan, 2010; Battista, 1983; Schwartz et al., 1993). This is a striking observation, especially because these Grade 8 students were exposed to temperature and electron contexts in their instruction (indicated by their teacher), lived in the Midwestern region of the United States where the temperatures reach negative temperatures, and were considered mathematically advanced by their teacher. Temperature is a context that supports thinking about and using the integers with Relativity or Translation conceptual models because of the relative scales and linear movement typically used in both. Electron charge also may support thinking about electrons with Relativity because an atom can have a charge with infinitely many different combinations of protons and electrons (e.g., -2 + 3 = +1, -5 + 6 = +1). It is noteworthy to compare the neglected use of the temperature and electron charge contexts to the diminutive use of the Relativity or Translation conceptual models by the students in this study. Similarly, the context of electron charge may promote the thinking about the integers with the Counterbalance conceptual model. Electrons were not used as a context once, yet the students drew upon the Counterbalance conceptual model twice with unconventional, realistic contexts (i.e., good deeds/bad deeds, black and white jerseys on a football team), without having to change the number sentences. The use of conceptual models for integer addition and subtraction in tandem with unconventional contexts supports the re-examination of the usefulness of conventionally advocated contexts. If students do not draw upon these contexts, we need a re-examination of conventionally used contexts to determine the extent in which using these contexts remain useful or even necessary.

The prevalence of the Bookkeeping model may be due to how the model naturally extends from whole number reasoning. With whole number addition and subtraction, there are not well-established conventional contexts like there are for integer operations. Any context that supports adding discrete objects together works for arithmetic with positive whole numbers (e.g., apples, pebbles, pencils, cupcakes). When students extend their thinking from positive whole numbers to using negative integers, they may modify their usual use of objects with "gains" or "losses." When students transition to using negatives, they can begin thinking about "losing" pencils or "wanting" pencils in opposing to "gaining" pencils or "having" pencils. While this context (losing or wanting pencils) may have limitations and constraints with some number sentences (e.g., 2 – -7), the context can work well with addition number sentences (e.g., -7 + 2, "I wanted seven pencils, obtain 2 pencils, and now only want 5 pencils."). Future work could explore the potential of understanding how students' use of unconventional contexts and conceptual models develop. Additionally, work could explore relationships between number sentences and context use and how contexts are instructionally useful.

Affordances of the conceptual models: Use in advanced mathematics. Understanding mathematical thinking and uses of the integers as students apply them to contexts may help students in more advanced mathematical work. The following discussion will unpack some of the possible affordances of the conceptual models for integer addition and subtraction in advanced mathematics.

Bookkeeping. The ideas afforded by the Bookkeeping conceptual model extend to mathematical uses beyond the traditional conventions of money, to thinking about numbers integers as gains and losses, which can be used to solve other problems beyond integer addition and subtraction. For example, consider computing means in statistics. If the mean of a data set is a certain amount that is given and an unknown element is added to that data set, then one might want to determine what that element needs to be for the mean to remain the same, for the mean to increase by a certain amount, or for the mean to decrease by a certain amount. This type of problem, and others, can be solved with a Bookkeeping, or a gains and losses, perspective.

Counterbalance. Negative numbers are used to conceptualize negative areas between the x-axis and a curve that lies below the x-axis in calculus. When integrating in calculus or computing Riemann sums in analysis, there is a counterbalancing of positive and negative areas between curves and the x-axis. If the curve is above the x-axis, the area is positive, and if the curve is below the x-axis, the area is negative. When the definite integral is 0, we know that the areas above the x-axis are equivalent to the areas below the x-axis. Although the result of an integral may be zero (see Figure 3), the areas are still present, making use of the Counterbalance conceptual model.

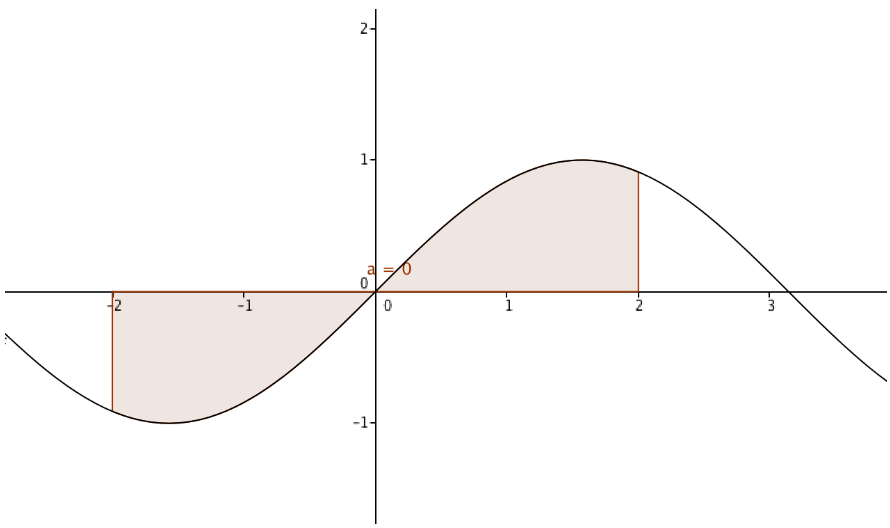
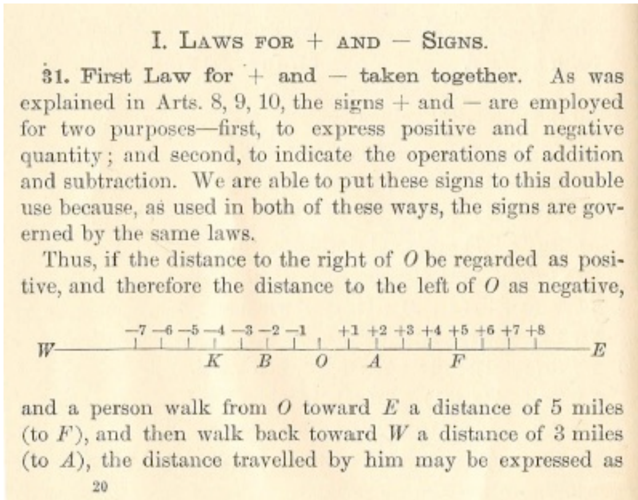


Figure 3. Example of positive and negative areas counterbalancing.

Relativity. Negative integers are found on the various axes as positions with relativity. Historically, before discussions about rules of operating with integers, the 19th century texts often discussed the nature of integers. For these authors, part of the nature of the integers was rooted in the relative nature of using the integers. For example, one of the few number line illustrations in these texts provided an arbitrary zero that highlighted the relative positions of the integers (see Figure 4). Rather than placing zero on the number line, the idea of relativity is emphasized with the referent as the point “O.” All of the integers are measurements from point, "O." Many of the authors of United States 19th-century arithmetic and algebra texts maintained that the use of integers is relative.



I. Laws for + and − Signs.

31. First Law for + and − taken together. As was explained in Arts. 8, 9, 10, the signs + and − are employed for two purposes—first, to express positive and negative quantity; and second, to indicate the operations of addition and subtraction. We are able to put these signs to this double use because, as used in both of these ways, the signs are governed by the same laws.

Thus, if the distance to the right of *O* be regarded as positive, and therefore the distance to the left of *O* as negative,

and a person walk from *O* toward *E* a distance of 5 miles (to *F*), and then walk back toward *W* a distance of 3 miles (to *A*), the distance travelled by him may be expressed as

20

Figure 4. A relative number line (Day & Thomson, 1843, p. 20).

Although the contexts of temperature and elevation are embedded in our current mathematics curriculum (NGA & CCSSO, 2010; Whitacre et al., 2011), ideas like relativity are absent and underdeveloped in the modern curriculum in the United States. Yet, this idea of the Relativity conceptual model has implications in the learning of more advanced mathematical topics, such as applying a relative position of the origin and Cartesian coordinate plane onto existing curves. Axes can be shifted and new planes, such as polar coordinates, can be introduced in order to obtain simpler equations for curves. The idea of axes and coordinate planes can be relative, just as the assignment of negative and positive numbers can be relative. For example, compare the graphs of $G(x) = x^2$ and $F(x) = (x-2)^2 + 3$ in Figure 5 (i.e., $G(x)$ is blue and $F(x)$ is purple). Both the graphs of $G(x)$ and $F(x)$ have equivalent shapes. $F(x)$ is a Translation of $G(x)$ two units to the right and three units up. However, instead of thinking of translating the graphs, we think of creating a new coordinate system. Lines a and c, which are the green lines in

Figure 5, could represent a new coordinate system and $F(x) = (x-2)^2 + 3$ could be redefined as $F(x') = (x')^2$ with this new coordinate system created with the new axes.

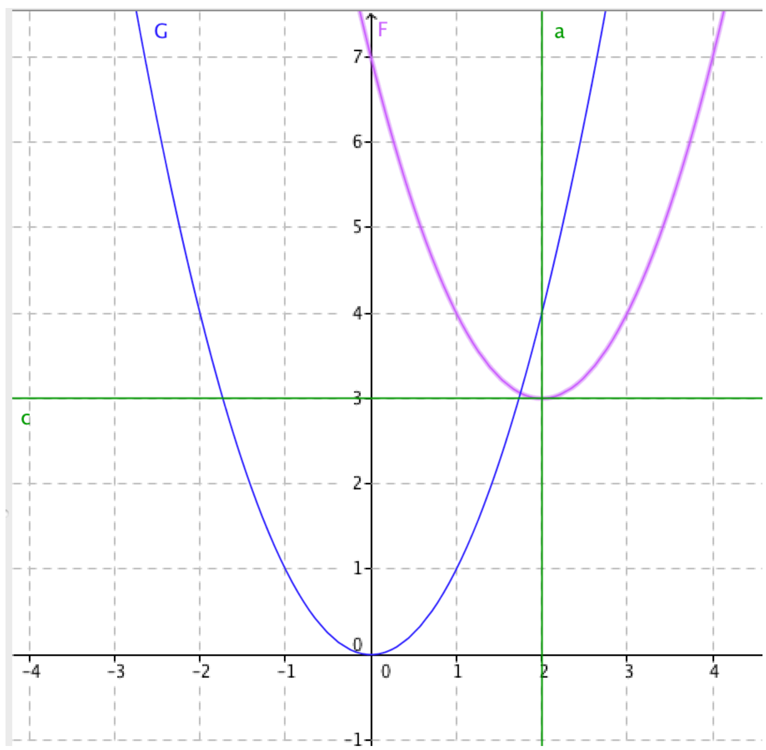


Figure 5. The use of Relativity here is demonstrated by the functions *F* and *G*, which are in the same family of quadratics. With a new coordinate system, *F* could be redefined as *G*.

How axes are used or even what coordinate system (e.g., Cartesian, polar) is selected are examples that support the use of the Relativity conceptual model in advanced mathematics.

Translation. Integers are used as transformations in algebra, geometry, trigonometry, and calculus when translating points, shapes, and curves. Eventually students learn that these transformations can be expressed as vectors, which they may use in physics and in the sciences. The integers also serve as both scalars and vectors. As students deal with geometric and algebraic transformations, they have to coordinate the use of integers both as scalars and as vectors. As students progress in more advanced mathematics they will eventually need to learn how using a negative integer as a scalar is different than using integers as vectors. Early experiences with utilizing the Translation conceptual model, or treating the integers as vectors, may help students as they progress mathematically.

Future Research

The major finding of this study is the descriptions of the conceptual models and the students' use of them, which describe ways that students think mathematically about integer addition and subtraction when posing stories. Future investigations, where children are worked with intensively and extensively, could use this set of models. The integrity of these models could be investigated in symbolic settings (e.g., as children solve $-2 + \square = 7$) since the models were developed from the stories that the students posed. Furthermore, future investigations could include how these models describe thinking over extended time or how models work with integer multiplication and division. Next steps in research could also be taken by employing this type of research to younger children and with a larger sample of children. This is preliminary work on generating and describing conceptual models for integer addition and subtraction. Future research should investigate the interconnectivity of these conceptual models, how the various conceptual models are related, and how learning may develop from these models (Thompson, 1982). Clement (2000) pointed to the importance of generative work on models and using these descriptions as a basis for generating more detailed descriptions of the models, or even other models, is needed.

Conclusion

The conceptual models of integer addition and subtraction provide a useful framework for understanding student thinking about integers addition and subtraction and contexts. This is important for the field for (a) understanding student thinking, (b) providing insight into the teaching and learning of the integers in the realm of contexts, and (c) offering a mathematical lens for using and thinking about contexts and negative numbers. The findings from this research are critical in extending the previous work on student thinking about integers because they describe thinking about and using the integers in relationship to contexts.

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