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Grade 5 Children's Drawings for Integer Addition and Subtraction Open Number Sentences

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1. Introduction

When solving addition and subtraction problems with positive integers, children often use strategies that incorporate drawings (Carpenter, Fennema, Franke, Levi, & Empson, 2015). These drawings often directly model problem situations or illustrate counting strategies as they make sense of the problems. The use of drawings might also resurface as children cope with changes in problem type (e.g., part-part-whole problems compared to separate problems, ☐ – 2 = 5 compared to 7 – ☐ = 5) or larger number sizes (Carpenter et al., 2015). Transitioning from whole number to negative numbers (Bofferding & Wessman-Enzinger, 2017) may similarly affect the types of drawings children create. Young children produce sophisticated drawings for negatives integers as they transition from whole number reasoning, including partitioned numbers lines and invented notation for negative integers (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014).

Beyond drawings, there is a significant amount research highlighting different instructional models for the teaching and learning of integers (e.g., Janvier, 1985; Liebeck, 1990; Saxe, Diakow, & Gearhart, 2013); and, there are various descriptions of how one may think about the broad notion of integers (Gallardo, 2002), the order, value and directed magnitudes of integers (Bofferding, 2014), and addition and subtraction about integers (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016; Whitacre et al., 2017). We know that children often re-organize their previous thinking in new ways (Steffe, 1992); and, as children reason with integers they often make connections from whole number reasoning to integer reasoning (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bofferding, 2014; Bofferding & Wessman-Enzinger, 2017).

Considering the transition from whole number to integers, what do children's drawings look like as they illustrate their thinking and learning? Although there are descriptions of children's thinking (e.g., Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bishop et al., 2016), we know little about the specific drawings, markings, and notation that children employ as they transition from using positive integers to negative integers. The literature, for example, has highlighted sophisticated drawings like invented notation for integers (Bishop et al., 2011) and use of number line (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). But, we lack frameworks that describe the nuances and uniqueness of these different types of drawings that children produce for integer addition and subtraction. Yet, drawings provide insight into children's thinking about integers, just as much as their verbal contributions (Sfard, 2008). For these reasons, descriptions of an inaugural framework for these types of drawings produced are introduced in this article, highlighting the novel and sophisticated work of the children from this study. In addition to a framework, how these drawings may change over time is described.

2. Background

2.1 Frameworks on children's thinking about integers—an evolutionary tale

The development of the various frameworks describing thinking about integers is an evolutionary tale. A contemporary taxonomy of mental models for order, value, and directed magnitude by Bofferding (2014), for example, extends the inaugural descriptions of divided number line and continuous number line mental models (Peled, Mukhopadhyay, & Resnick, 1989; Peled, 1991). For addition and subtraction, Bishop, Lamb, Philipp, Whitacre, Schappelle, and Lewis (2014) first put forth descriptions of different ways of reasoning about integer addition and subtraction (i.e., order, magnitude, logical necessity, computation, limited). Then, Bishop et al. (2016) modified this framework (e.g., removing magnitude reasoning and including analogy reasoning). Similarly, a set of conceptual models (Wessman-Enzinger & Mooney, 2014; i.e., bookkeeping, counterbalance, relativity, translation) evolved that includes more robust descriptions and other categories (Wessman-Enzinger, 2015; e.g., proceduralization, algebraic reasoning, and analogy).

These frameworks illustrate children's thinking about integers and provide evidence of evolution and development. Although these frameworks highlight the sophisticated accomplishments of children, these frameworks do not provide explicit insight in the constructed and created drawings of children. Such a framework about drawings, is needed in order provide further insight into these existing frameworks—the next step in the evolution and development of integer frameworks.

2.2 Integer instructional models

Because children often draw objects as they solve whole number addition and subtraction problems (Carpenter et al., 2015), it seems likely that children could potentially invent or construct drawings similar to a cancellation model (e.g., Battista, 1983; Koukkoufis & Williams, 2006; Linchevski & Williams, 1999) used in integer instruction. One way of using a cancellations model is representing the negative integers with red chips and positive integers with black chips (Liebeck, 1990). When employing two-colored chips within a cancellation model, a negative integer, -*n*, requires that *n* objects are physically present and countable in the model. Thus, representing -*n* by extension requires that each countable object represents -1, using *n* red chips. A challenge with this model, is that “zero pairs” (i.e., a chip representing a positive integer and a chip representing a negative integer) must be added to the set of chips in order use the model with number sentences like 2 – -5 = ☐. Adding in zero pairs of chips may not be an intuitive aspect of the model (Vig, Murray, & Star, 2014); children will not likely construct this type of drawing for problems like 2 – -5 = ☐.

Children have illustrated creative ability in inventing notation and drawing number lines (Bishop et al., 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014); but more learning about their specific drawings is needed, especially in relationship to number line models used in integer instruction (e.g., Herbst, 1997; Saxe et al., 2013). Although subtraction number sentences (e.g., 2 – -5 = ☐) could be represented as the distance between two integers (Bofferding & Wessman-Enzinger, 2017; Wessman-Enzinger & Salem, 2018), instructional models for the number line often advocate for other “rules” when using integers (Peled & Carraher, 2008). When considering 2 – -5 = ☐, for instance, subtracting a negative integer from a positive integer with a number line model may include a set of contrived rules: start at 2, turn around, and walk backwards 5 units. It is not likely that children will invent such a contrived use of the number line.

Because these types of instructional models for integer addition and subtraction are somewhat contrived (e.g., adding in zero pairs and turning around and walking backwards are not inherently intuitive), a bottom-up approach is pivotal for understanding the drawings that children produce when these typical instructional models are not provided to them.

Gaining insight into children's drawings about integers may influence future instructional support that facilitates learning or conceptual change (Bofferding, 2014). Investigating children's drawings helps provide insight into affordances and hindrances of integer instructional models (Vig et al., 2014) that are used for integer instruction by providing insight into alternative instructional models that may exist, imbedded in the children's creations. Therefore, this article focuses on the drawings created by children, without prior formal school instruction with negatives in school, as they solved integer addition and subtraction open number sentences. Investigations into their drawings provide more insight into these integer instructional models and how they may potentially be used in the future.

3. Theoretical Perspective

3.1 Learner-generated drawings

Learner-generated drawings offer potential in understanding children's thinking about integer addition and subtraction further (van Meter, Aleksic, Schwartz, & Garner, 2006). An abundance of literature about learner-generated drawings may be found in other domains, like science education (e.g., Schwamborn, Mayer, Thillmann, Leopold, & Leutner, 2010; van Meter et al., 2006; Zhang & Linn, 2011, 2013); investigations in mathematics education have highlighted the possibility that students may find certain visuals or illustrations meaningful (Dewolf, Van Dooren, Cimen, & Verschaffel, 2014; Elia & Philippou, 2004; Hoogland, Pepin, Bakker, de Koning, & Gravenmeijer, 2016), but we lack frameworks for integer drawings constructed specifically by learners.

Learner-generated drawings (van Meter et al., 2006) are differentiated from illustrations, pictures, or visuals (e.g., Dewolf et al., 2014; Elia & Philippou, 2004) in that the learner, without a previously provided illustration, constructs the drawings. While this may seem like a nuanced differentiation, it is an important distinction when thinking about mathematical objects that children create themselves.

van Meter and Garner (2005) provide a definition of drawing that works for mathematics. van Meter and Garner delineate learner-generated drawings as:

> pictorial representations (a) that are intentionally constructed to meet learning goals, (b) that are meant to depict represented objects accurately and, (c) for which the learner is primarily responsible for construction and/or final appearance. (p. 290)

This definition supports children or students as the primary agent for constructing drawings where the final product is solely dependent on the child or student, which is different than existing frameworks for illustrations and visuals in mathematics education (Elia & Philippou, 2004). This construction process is often done where the student makes the drawing free hand with only paper and pencil as tools (van Meter & Garner, 2005). This definition of learner-generated drawing works for mathematical drawings because children often produce drawings when solving problems and their drawings often depict mathematical objects or contextual situations (Carpenter et al., 2015).

3.2 Types of learner-generated drawings

van Meter and Garner (2005) identify three different types of learner-generated drawings: strategic, representative, and constructive drawings. One type of learner-generated drawing that van Meter and Garner (2005) distinguish is *strategic drawings*. Student used strategic drawings for achieving goals and organizing knowledge. Learner-generated drawings, as a strategy, improve learning (e.g., van Meter et al., 2006).

The second type of learner-generated drawing that van Meter and Garner (2005) describe is *representative drawings*. Learners may also use representative drawings for generating ideas. Within the literature on learner-generated drawings, representative drawings are often used in reading strategies (e.g., Caldwell & Moore, 1991). However, mathematics is full of representations (e.g., number line) and learner-generated drawings may potentially highlight potential representations of children's mathematics or mathematical objects, even if inaccurate.

The third type of learner-generated drawing that van Meter and Garner (2005) report is *constructive drawings*. Learners create constructive drawings and these drawings involve "constructive learning processes that engage nonverbal modalities and requires integration" (van Meter & Garner, 2005, p. 288). If learners have not had previous instructional experiences or formal school instruction with integers, then the drawings that children produce as they make sense of integer operations will be constructive drawings.

4. Method

4.1 Background of study

Three Grade 5 children, each 10 years old, from a rural Midwest school in the United States participated in a microgenetic study (Seigler & Crowley, 1991) imbedded within a 12-week teaching experiment (Steffe & Thompson, 2000). The teaching experiment, reported elsewhere (Wessman-Enzinger, 2015), consisted of group sessions centered on providing contextual situations for integer addition and subtraction. The embedded microgenetic study included four structured individual open number sentence sessions with each child across the 12-week teaching experiment.

The individual open number sentence sessions of the microgenetic study occurred across the teaching experiment for understanding the children's thinking individually and exploring changes in how the children solved integer addition and subtraction across the 12-weeks. These sessions of the imbedded microgenetic study represent the only place in the study when the children solved only symbolic problems (e.g., -2 + ☐ = 5). Because the children were not provided manipulatives or shown images at any time during the teaching experiment or microgenetic study, they constructed a wealth of learner-generated drawings. The microgenentic study reported in this article draws on the robust data of learner-generated drawings from these individual open number sentence sessions.

Although the children produced drawings throughout the teaching experiment, these drawings are excluded from this study for two reasons. First, the children solved only contextual problems during the teaching experiment group sessions. The drawings produced when solving contextual problems seemed directly linked to that specific context. For example, when the children solved a problem in the context of temperature they sometimes drew a physical thermometer. Whereas, when they solved a problem about debt, they did not draw a thermometer. Focusing on the drawings from individual open number sentence sessions provided opportunity for investigating change—an important aspect of a microgenetic design (Seigler & Crowley, 1991)—as the children solved similar symbolic problems, rather than the varying contextual problems imbedded in the teaching experiment. Second, in the teaching experiment, the children sometimes used each other’s drawings. The individual open number sessions provided a unique space for examining what the children constructed individually.

4.2 Participants

Three children (Alice, Jace, Kim) participated because the focus of both microgenetic (Siegler & Crowley, 1991) and teaching experiment methodology is on extended time across the 12-weeks, rather than sample size of children (Steffe & Thompson, 2000; Thompson & Dreyfus, 1988). These Grade 5 children constituted ideal participants because they did not have prior instructional experiences with integer operations and volunteered in their free time during the school day. Also, Grade 5 children are close in age to *Common Core State Standards* recommendations, which recommend integer operations in Grade 7 (National Governors Association & Council of Chief State School Officers, 2010). Consequently, the first open number sentence session represents their first time engaging in a formal experience with integer addition and subtraction.

4.3 Individual open number sentence sessions

In four individual open number sentences sessions across the 12-weeks, the children solved 20, 23, 25, and 25 integer addition and subtraction open number sentences, respectively (see Figure 1). Within the individual sessions, the children solved twenty open number sentences of the same types (e.g., $-20 + 15 = \square$ and $-16 + 4 = \square$ are different than $1 – \square = 3$ and $4 – \square = 6$). The number sentence $-20 + 15 = \square$, for instance, is considered the same problem type as $-16 + 4 = \square$ because in $-a + b = \square$, $a > b > 0$ (Murray, 1985). The children solved more problems in individual sessions 3 and 4 because they solved more problems in later sessions during the fixed 60-minute timeframe.

Individual Open Number Sentence Session 1	Individual Open Number Sentence Session 2	Individual Open Number Sentence Session 3	Individual Open Number Sentence Session 4
-20 + 15 = □	-16 + 4 = □	-18 + 12 = □	-20 + 15 = □
12 + -16 = □	20 + -33 = □	15 + -24 = □	12 + -16 = □
-4 + □ = 10	-6 + □ = 15	-3 + □ = 14	-4 + □ = 10
-7 + □ = -2	-6 + □ = -1	-9 + □ = -3	-7 + □ = -2
□ + -3 = 7	□ + -2 = 17	□ + -4 = 13	□ + -3 = 7
□ + 13 = -5	□ + 19 = -4	□ + 25 = -2	□ + 13 = -5
-8 + -7 = □	-12 + -5 = □	-17 + -6 = □	-8 + -7 = □
-2 + □ = -10	-4 + □ = -19	-5 + □ = -21	-2 + □ = -10
□ + -9 = -16	□ + -9 = -21	□ + -9 = -17	□ + -9 = -16
10 – 12 = □	5 – 9 = □	12 – 18 = □	10 – 12 = □
1 – □ = 3	4 – □ = 6	3 – □ = 4	1 – □ = 3
-5 – 4 = □	-9 – 8 = □	-5 – 3 = □	-5 – 4 = □
2 – -3 = □	3 – -4 = □	1 – -3 = □	2 – -3 = □
-1 – □ = 8	-2 – □ = 9	-2 – □ = 10	-1 – □ = 8
2 – □ = -10	6 – □ = -10	4 – □ = -12	2 – □ = -10
□ – -1 = 6	□ – -1 = 4	□ – -2 = 5	□ – -1 = 6
□ – 8 = -5	□ – 9 = -3	□ – 6 = -2	□ – 8 = -5
-15 – -4 = □	-11 – -2 = □	-12 – -4 = □	-15 – -4 = □
-12 – □ = -13	-15 – □ = -16	-10 – □ = -11	-12 – □ = -13
	□ – -3 = 2	□ – -3 = 1	□ – -2 = 1
	□ – -4 = 0	□ – -5 = 0	□ – -3 = 0
	12 + □ = 8	15 + □ = 9	17 + □ = 8
		8 + □ = -5	6 + □ = -2
		□ + 2 = 0	□ + 4 = 0
		-4 – 10 = □	-2 – 8 = □

Figure 1. The open number sentences the children solved during the Individual Open Number Sentence Sessions.

Children solved each of the integer addition and subtraction open number sentences on a single sheet of paper. The children did not have manipulatives available for the entire study, but did have a box of markers. The children explained their reasoning for solving the open number sentences and the teacher-researcher did not tell them if they were right or wrong or how to solve the problems at any point. Drawings produced by the children during the individual sessions, whether a number sentence or illustration, constituted the units of data, constituting 93 units per child and 279 total units of data for this investigation.

4.4 Research questions

The following research questions emerged from this production of drawings during the individual sessions:

1. What types of drawings did the Grade 5 children produce as they solved integer addition and subtraction open number sentences?
2. In what ways did the children's use of drawings change across the individual sessions?

4.5 Data analysis

Data was examined using constant comparative methods (Merriam, 1998), beginning with descriptions of drawings of what children draw for whole numbers. It was expected that children might draw objects as they do for whole numbers (Carpenter et al., 2015), number lines (Saxe et al., 2013), empty number lines (Verschaffel, Greer, & De Corte, 2007), or number paths[1] (Bofferding & Farmer, 2018). Units of data were examined and then sorted into similar groups. For example, if units of data had number lines they were put together or if units of data included tallies they were grouped together. Then, groups were examined and sorted further. The different types of grouping of the learner-generated drawings created by the three Grade 5 children during these individual sessions became the categories of learner-generated drawings. Once the similar groupings of drawings were sorted and named, definitions were developed and then used as codes. The definitions incorporated language about negative integers since the previous literature on children's drawings focused on whole numbers. The categories of types drawings are described in the first results section.

Once the set of definitions and codes were developed, the units of the data were coded with these categories of drawings with several passes. Counts and percentages

[1] Number paths are a linear set of boxes structured liked a number line, but without the scale and tick marks. Instead, integers are inside of the boxes that form linear "paths" of numbers.

provided of these codes provided insight into the use of drawings for making sense of how the drawing productions changed over time. Although there are only three children, there are 279 units of data; counts and percentages facilitate making sense of this qualitative data (Miles & Huberman, 1994). Then, the correctness of the children's solutions of the open number sentences, paired with the drawings, provides further insight into the types of changes beyond frequency of production of the drawings. These changes in drawing use are described in the second results section.

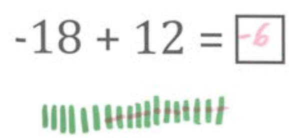
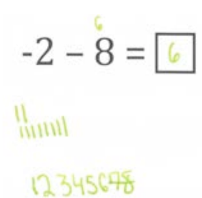
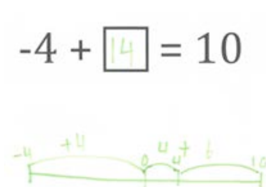
A second researcher coded 25 units of randomly selected data for each of the children, for a total of 75 units of data, or about 1/4 of the total data, for reliability. The second researcher and author agreed 88% of the time (66 out of 75 codes). One disagreement centered on coding Number Sequence versus Empty Number Line. This disagreement occurred because no line, or scale, was drawn; Empty Number Line was agreed upon because there were tic marks, number paired with the tic marks, and intentional spacing. Sometimes a child changed their solution (e.g., from -6 to 6) and crossed off a negative sign; this resulted in disagreements when coding for Sign Emphasis. These disagreements were discussed and negotiated as not a Sign Emphasis because we considered crossing off of a previous answer as not emphasizing a sign. All other disagreements were oversights when a unit of data had 2 or more different types of drawings in them. For example, one of the coders over looked two drawings in one unit of data coded as both Double Set of Objects and Vertical Number Sentence. That coder overlooked Double Set of Object in that unit. For this reason, after meeting with the second researcher and comparing codes, a final pass through the data was made again, specifically looking for this particular coding error.

5. Types of Learner-Generated Integer Drawings

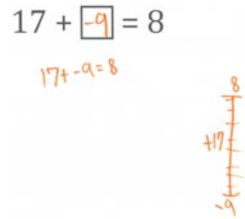
The different types of drawings that emerged included: Single Set of Objects, Double Set of Objects, Number Sequences, Empty Number Lines, Number Lines, Horizontal and Vertical Number Sentences, Plus, Minus, or Negative Sign Emphasis, Answer in Box Only, and Blank. These drawings, which are illustrated in Table 1, will be described individually and in more depth next.

Table 1

Different Learner-Generated Drawings Produced by Grade 5 Children

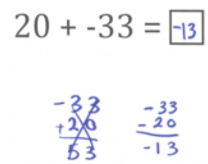
Type of Drawing	Example of Drawing
Discrete Objects	
Single Set of Objects: There is one set of discrete objects. These objects are added onto other objects, removed from a set of existing objects, or are used alone for counting on.	
Double Set of Objects: There are two sets of discrete objects. The two sets of objects are used in comparison to either other or the two sets of objects are used as different parts of a whole collection of objects.	
Transition from Objects to Number Lines	
Number Sequence: Numbers are used in an ordered manner or list. (Example includes a Double Set of Objects and a Number Sequence.)	
Continuous Number Lines	
Empty Number Line: There is a segment of a number line that does not use equipartitioning, but numbers are listed on the number line. The distances on the Empty Number Line may be highlighted.	

Number Line: There is a segment of a number line that attempts equipartitioning.
(Example includes a Number Line and a Horizontal Number Sentence.)

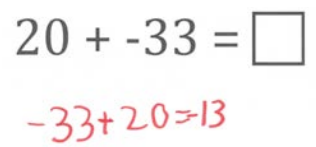


Number Sentences

Vertical Number Sentence: The number sentence is re-expressed as a number sentence written vertically.

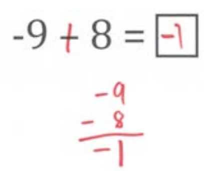


Horizontal Number Sentence: The number sentence is re-expressed as a number sentence written horizontally.

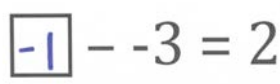


Other

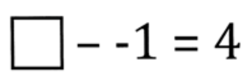
Plus, Minus, or Negative Sign Emphasis: The plus, minus sign, or negative sign is crossed off or circle or re-written.
(Example also includes a Vertical Number Sentence.)



Answer in Box Only: The only drawing produced was a solution in the box.



Blank: There is no drawing produced to solve the problem. Or, there is no drawing produced because the problem was not solved.



5.1 Single Set of Objects

One of the drawings included Single Set of Objects. This type of drawing consisted of one set of discrete objects. This drawing produced included objects added onto a singular set of objects, objects removed from an existing set of singular objects, or a set of objects used alone for counting on.

Of the three participants, Alice used Single Set of Objects most often (22%, 20/93 instances). Although Kim never produced a Single Set of Objects Drawing during the Open Number Sentence Individual Sessions, Jace produced a Single Set of Objects drawing once out of 93 units (1%). He created his Single Set of Objects in one of the first sessions. Jace, however, used this only as his first attempt at solving 12 + -16 = ☐, then abandoned it. He then used a vertical number sentence 16 – 12 = -4 and markings on the plus and negative symbols. Overall, the children created Single Sets of Objects in 20 of the total 279 units (7%).

The children used Single Set of Objects in two different ways. One way of producing Single Set of Objects is by crossing off objects from a single set of objects. A second way of producing Single Set of Objects is counting on or drawing extra objects onto a Single Set of Objects. For crossing off a Single Set of Objects, Alice began by drawing an initial set of objects (e.g., boxes or tallies), which represented either a positive or negative integer. She then crossed some of the objects off (see, e.g., Figures 2 and 3). The objects crossed off either represented adding or subtracting a positive or negative integer.

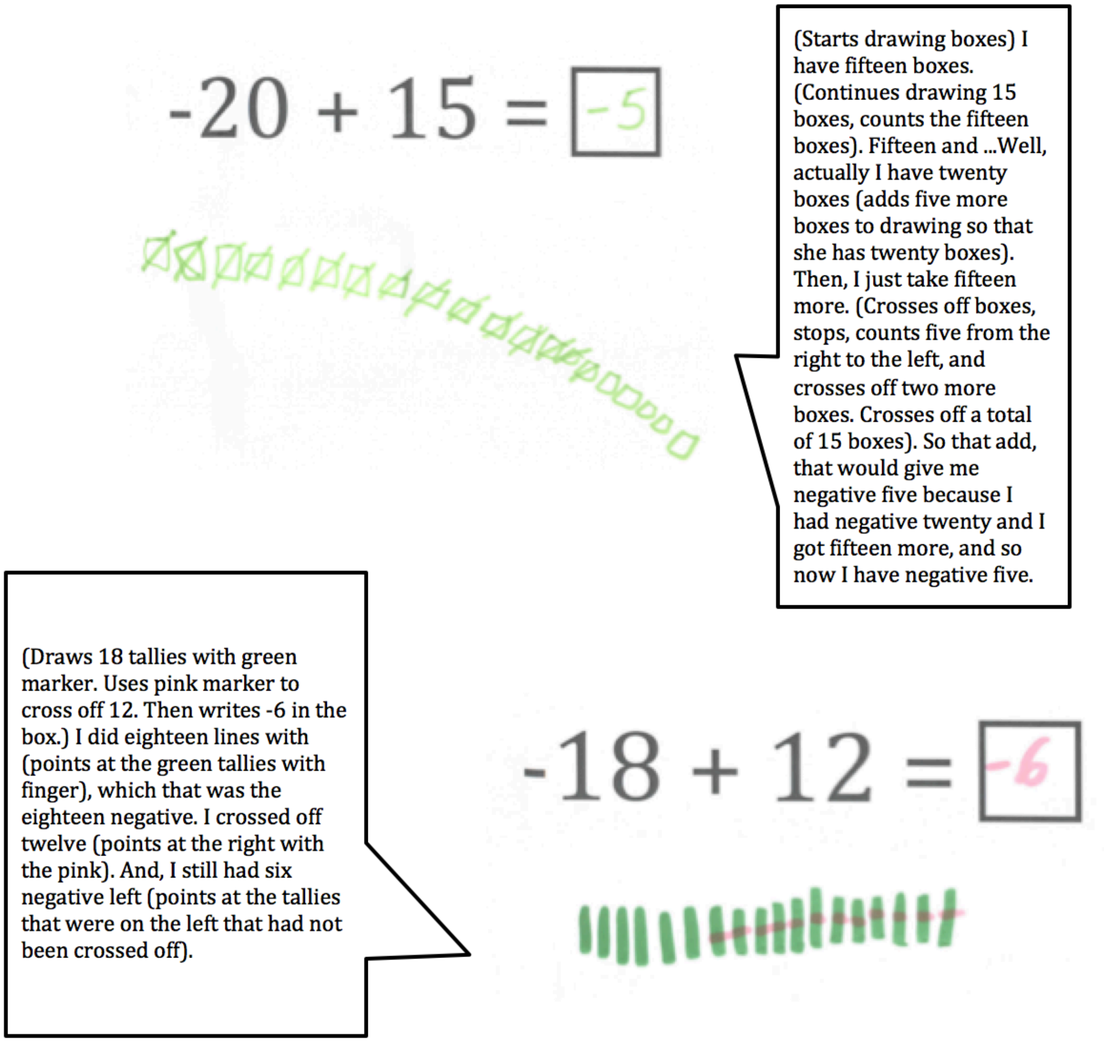


Figure 2. Alice created a Single Set of Objects and crossed off boxes when she solved -20 + 15 = ☐. Alice created a Single set of Objects and crossed off tallies from that Single Set of Objects as she solved -18 + 12 = ☐.

Alice's objects that she drew included either boxes or tallies for the Single Set of Objects (see Figure 2). She used either the boxes or the tallies to represent the negative integers. Then, she crossed off the amount of tallies that represented the positive integer being added. This type of drawing produced supported both correct and incorrect solutions (see, e.g., Figures 2 and 3).

Figure 3 shows an example of how Alice used Single Set of Objects in a way that produced an incorrect mathematical answer. Alice solved -9 – 8 = ☐ by representing -9 with 9 tallies and subtracting 8 by crossing off 8 tallies. Note that the drawing in Figure 3 is how Alice also routinely solved number sentences, like -9 + 8 = ☐, as demonstrated in Figure 2.

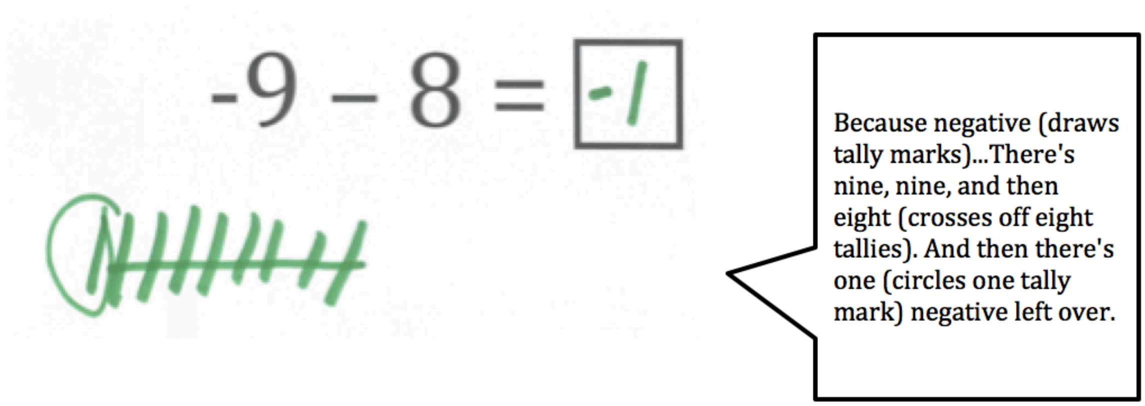


Figure 3. Alice created a Single Set of Objects and crossed off tallies when she solved -9 – 8 = ☐.

Alice and Jace also produced Single Set of Objects when she used the objects for counting on (see Figure 4). In Figure 4 Alice started with stating nine and drew boxes until she counted to sixteen. Then, she counted the boxes. She made an analogy to whole numbers, comparing 7 + 9 to -7 + -9, and wrote -7 in the box. Jace also produced a similar drawing (see Figure 4), but ended up crossing it off and producing a vertical number sentence instead. He added the markings on the plus sign and negative sign at the end of the session as he explained he reasoning further.

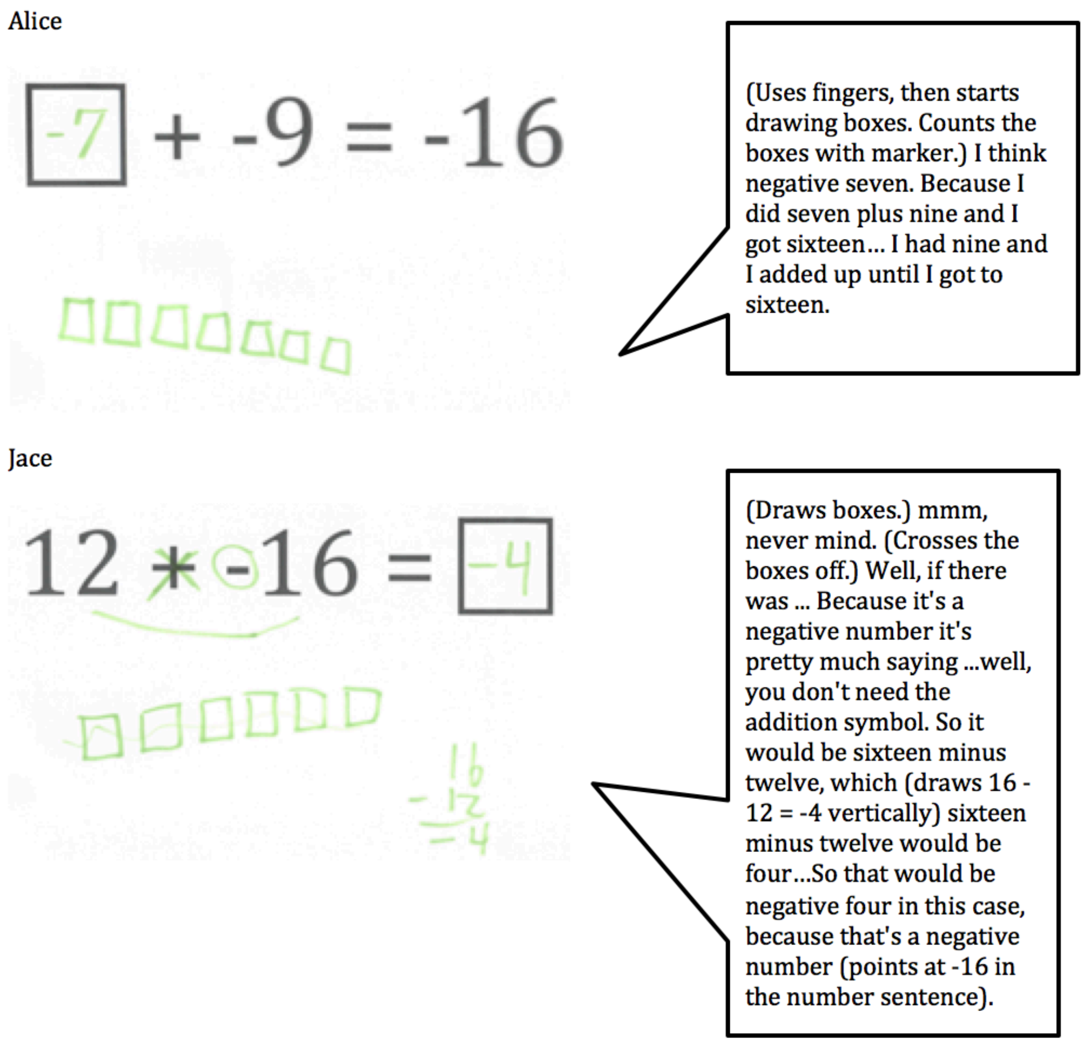


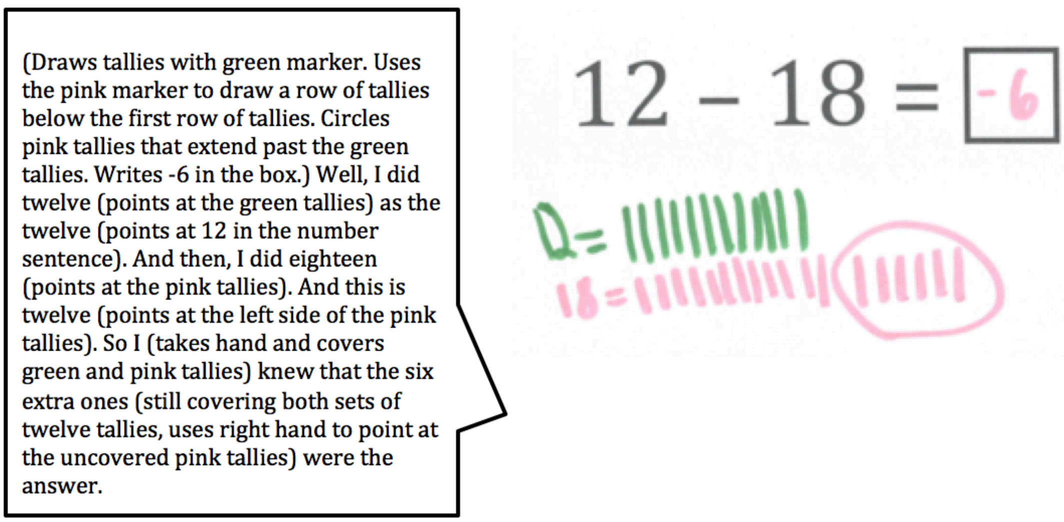
Figure 4. Alice created a Single Set of Objects by counting on from -9 when she solved ☐ + -9 = -16. Jace created a Single Set of Objects in an initial attempt when he solved 12 + -16 = ☐.

5.2 Double Set of Objects

One of the children, Alice, produced drawings that included a Double Set of Objects. The Double Set of Objects included two distinguishable sets of discrete objects. She used Double Set of Objects by comparing two different sets of objects or distinguishing two different parts of a set of objects.

Only one child, Alice, produced the Double Set of Objects. She used the Double Set of Objects drawing in 38 of the 93 drawings (41%). Neither Jace nor Kim produced

Double Set of Objects drawings. Because Alice is the only child that produced these types of drawings, there were 38 out of 279 total instances of this type of drawing (14%). Figures 5 and 6 illustrate two different types of Double Sets of Objects. Figure 5 includes a Double Set of Objects, for solving 12 – 18 = ☐, with twelve green tallies on top and eighteen pink tallies below. The leftover, circled tallies, not layered, illustrate the solution. Alice determined a solution of -6 with her drawing. These types of drawings that Alice produced included two layers or two separated groups of discrete objects.



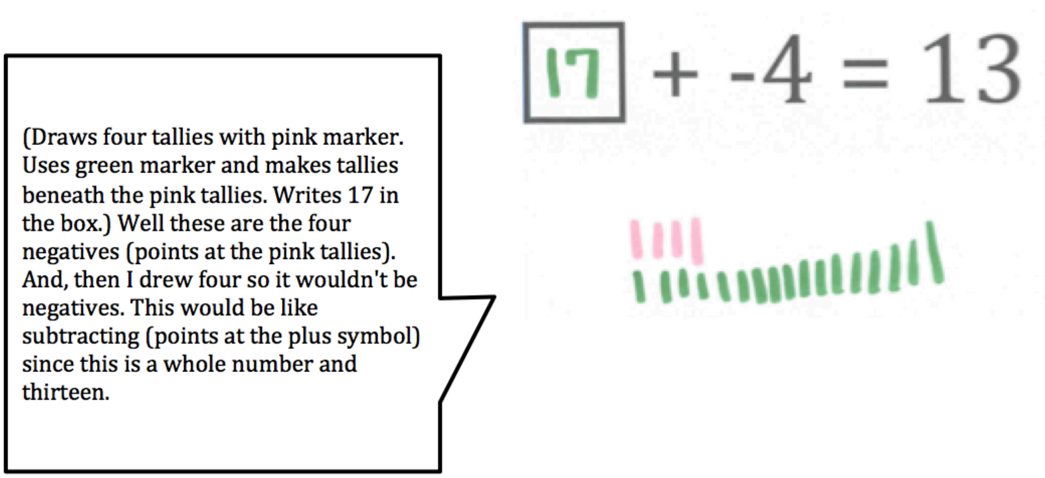


Figure 5. Alice drew two layers of tallies, demonstrating Double Set of Objects, when solving 12 – 18 = ☐, with a correct solution. She also drew Double Set of Objects, when she solved ☐ + -4 = 13, with a correct solution.

Alice used the Double Set of Objects with other open number sentence types, like ☐ + -4 = 13 in Figure 5. In Figure 5 for ☐ + -4 = 13, the pink tallies on the top represent negative four and the green tallies on the bottom represent 17. Alice drew green tallies until the tallies beyond the first layer totaled 13. Then, she counted all of the green tallies, determining the solution of 17.

Although Alice produced objects layered on top of each other (see Figure 5), she did not always produce Double Set of Objects in this way. Figure 6 highlights how she sometimes drew two sets of segregated objects, such as when she solved -4 + ☐ = 10. Alice represented -4 with 4 boxes aligned next to each other but segregated the second set into another layer. She added up all of the boxes obtaining 14. Alice described her drawing: “I did four for negative four (motions across four boxes), then I did how many I was adding a box for how many it would take to get me up to ten.” Unlike the drawings in Figures 5, this Double Set of Objects in Figure 6 is distinguished by two sets of objects side-by-side, horizontal but offset. Alice did not describe these boxes as orderings of numbers, like a number path (Bofferding, 2010). Rather, she described the amount of objects she had: “I did four for negative four (motions across four boxes).”

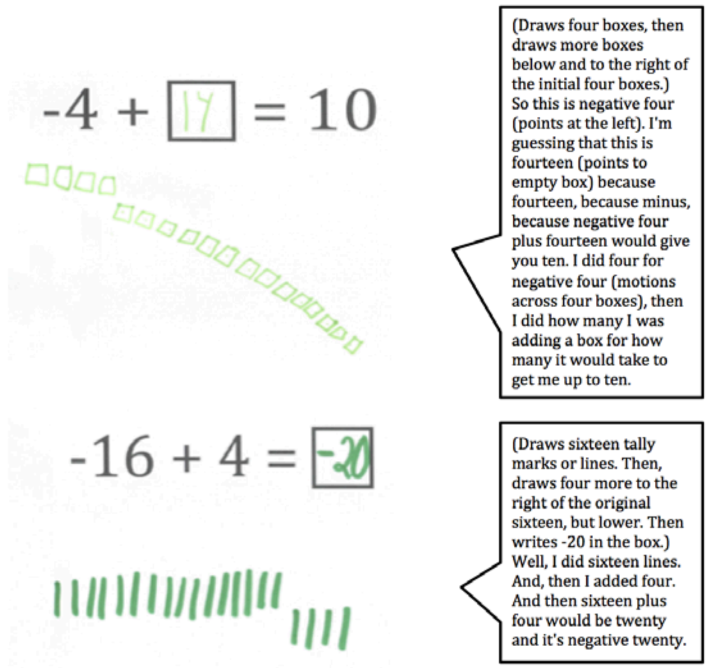


Figure 6. Alice drew two Segregated Layers of boxes, illustrating a Double Set of Objects drawing, when she solved -4 + ☐ = 10, with a correct solution. In a subsequent session, Alice drew two Segregated Layers of tallies, illustrating a Double Set of Objects drawing, when she solved -16 + 4 = ☐, with an incorrect solution.

Although this drawing is reminiscent of a number path with the horizontal boxes (Bofferding, 2010), it is without numbers in the boxes and Alice used this Double Set of Objects in a way that focused on the cardinality of the quantities rather than order (see Figure 6). This type of drawing did not always facilitate Alice in determining the correct solution. The Double Set of Objects, for example, illustrated in Figure 6 as she solved -16 + 4 = ☐, includes an incorrect mathematical solution. As Alice produced this drawing, she added up all of the tallies and found that there are twenty total tallies. She decided that since there were more of the tallies for -16 than the tallies for 4, the solution should be negative as well.

5.3 Number Sequence

The children produced drawings that included Number Sequences. The Number Sequence included numbers used in an ordered manner or list without the production of a line, scale, or line segment. Jace and Alice both drew Number Sequences when they found number sentences challenging.

The children did not produce Number Sequences often. Alice only produced Number Sequences twice out of 93 drawings (2%) and paired with other types of drawings. Similarly, Jace only produced 4 Number Sequence drawings of the 93 drawings (4%) and used these as his main drawing without other drawings. Kim did not create any Number Sequences during the Individual Open Number Sentence Sessions. In both of Alice's productions of Number Sequence, she also paired another type of drawing with it (see Figure 8). Overall, the children drew Number Sequences in 5 of the 279 units (2%).

Alice produced a Number Sequence, as well as a Double Set of Objects, when she solved the number sentence -2 – 8 = ☐. Alice first produced a Double Set of Objects and determined a solution of 6 for the number sentence -2 – 8 = ☐. Then she drew the Number Sequence and she connected her Number Sequence to her Double Set of Objects (see Figure 7). During her explanation of her solution, she drew a Number Sequence and crossed off two of the numbers, treating her Number Sequence like a Single Set of Objects.

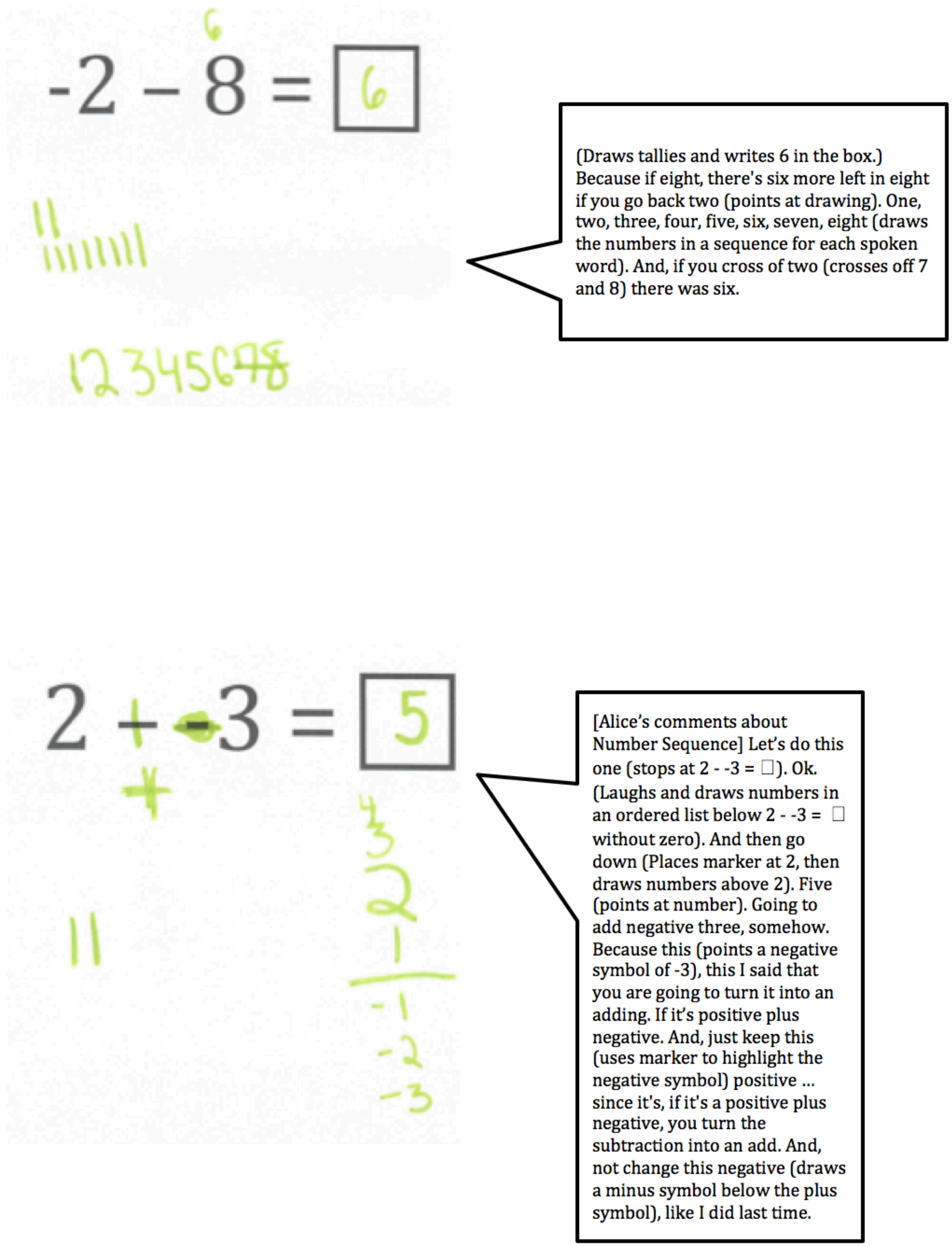


Figure 7. Alice drew both a Number Sequence and a Double Set of Objects as she solved -2 – 8 = ☐, with an incorrect solution; Alice produced a Number Sequence in addition to other drawings when she solved 2 – -3 = ☐, with a correct solution.

She drew a Number Sequence in Figure 7 when solving the open number sentences 2 – -3 = ☐ during the last individual session of the twelve weeks. To solve 2 – -3 = ☐, Alice first solved it by drawing upon a rule that the children developed during the group sessions, “Because it’s just like the last one. You do plus (changes minus sign to plus sign) and take that off (scratches off the negative symbol of -3). And, it just be like two plus three.” When asked why it worked, she drew both tallies and then a Number Sequence for justifying her rule. Although she drew a Number Sequence, she did not use it for solving the open number sentence or justifying the solution. Alice responded that she did not use her Number Sequence in Figure 7 for determining her answer.

In contrast, when Jace drew Number Sequences, he mostly produced these without other drawings (see, e.g., Figure 8). For example, in the third individual session Jace did not solve 3 – ☐ = 4, but he drew a Number Sequence horizontally with negatives on the right as he worked on solving the number sentence, 3 – ☐ = 4 (see Figure 8). But, when Jace solved ☐ – -2 = 1 in the last individual session, he created a Number Sequence by drawing numbers vertically and ordered (1, 0, -1). He used this ordered list of integers, counting two backwards from -1 to 1, determining a solution of -1 (see Figure 8).

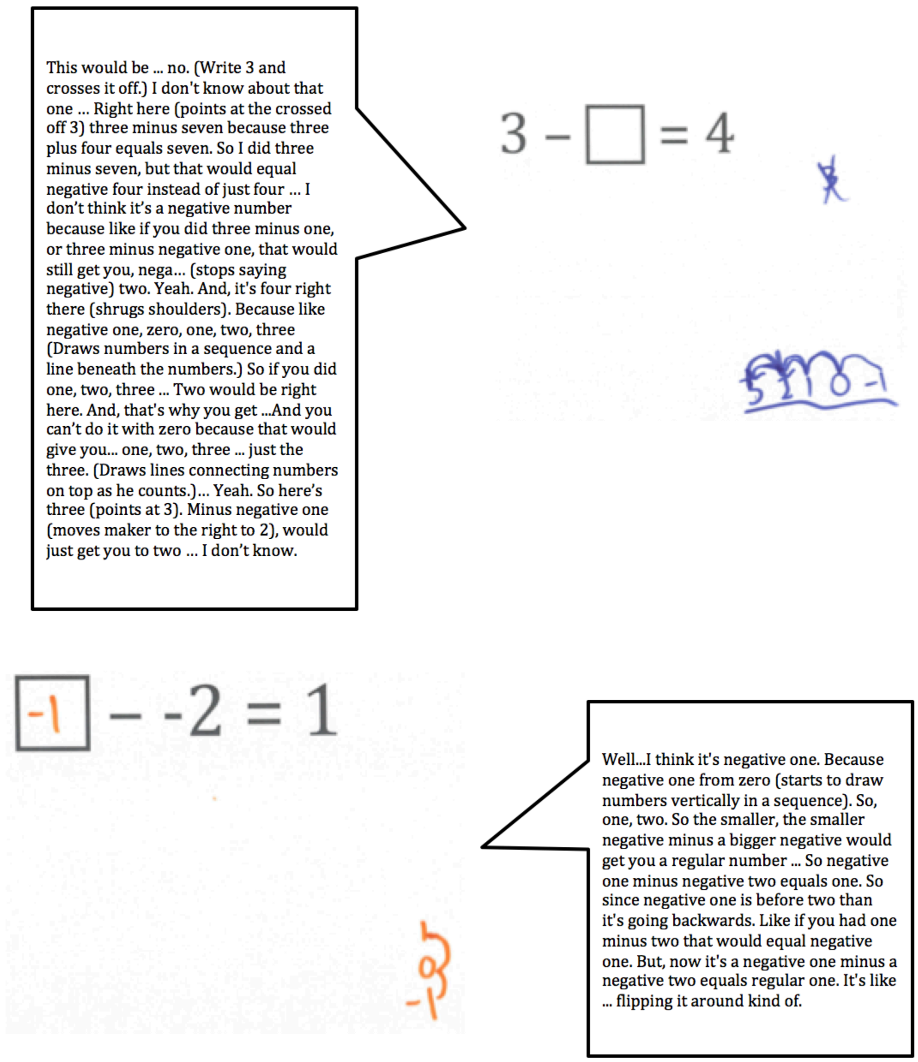


Figure 8. Jace drew a Number Sequence as he worked on solving 3 – ☐ = 4, with no solution determined in the third individual session. Then, Jace drew a Number Sequence in the last individual session when he solved ☐ – -2 = 1 and obtained the correct solution.

5.4 Empty Number Line

The children produced drawings that included Empty Number Lines. An Empty Number Line included a segment of a number line, not including equipartitioning. The numbers are listed on a number line, line segment, or scale. Empty Number Line sometimes included highlighted distances with arcs (see, e.g., Figures 9 and 10) and other times not (see, e.g., Figure 11).

Jace produced 6 Empty Number Line drawings out of 93 units (6%). Kim produced 5 Empty Number Line drawings out of 93 units (5%). And, Alice did not create any Empty Number Line drawings in the Open Number Sentence Individual Sessions. Overall, the children drew 11 Empty Number Line drawings out of the 279 units (4%).

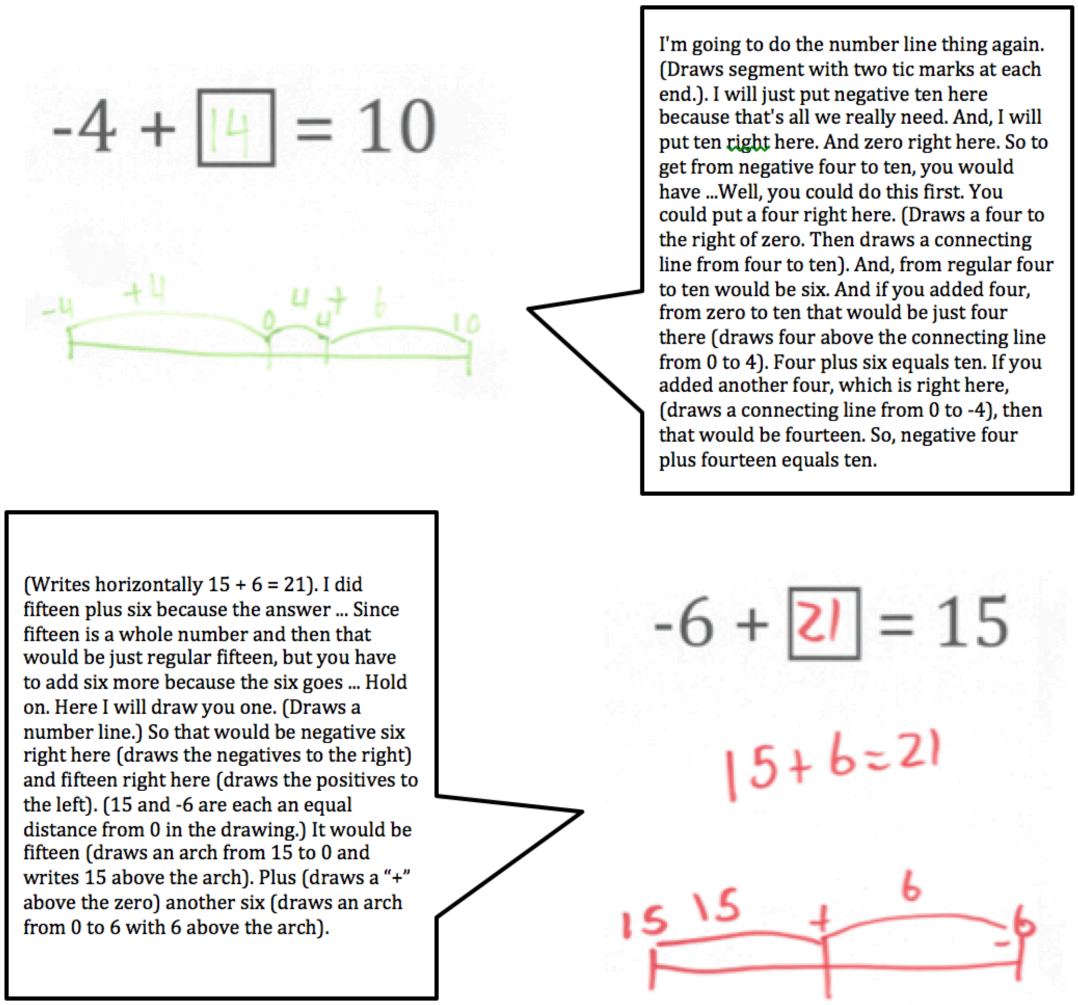


Figure 9. Jace drew two horizontal Empty Number Lines in different sessions, with negative numbers on the left and positive numbers on the right, when he solved -4 + ☐ = 10 and with negative numbers on the right and positive numbers on the left, when he solved -6 + ☐ = 15. The top figure is from the first individual session and the second Empty number is from the second individual session.

The children demonstrated flexibility with the positioning of the Empty Number Line—whether horizontal (e.g., Figure 9) or vertical (e.g., Figure 10). Similarly, the children also placed the negative integers flexibly on the right or left or top or bottom of the Empty Number Line. Jace, for instance, used the negatives on the left side of an Empty Number Line drawing while solving one problem and then used the negatives on the right side of an Empty Number Line drawing while solving another problem (see Figure 9).

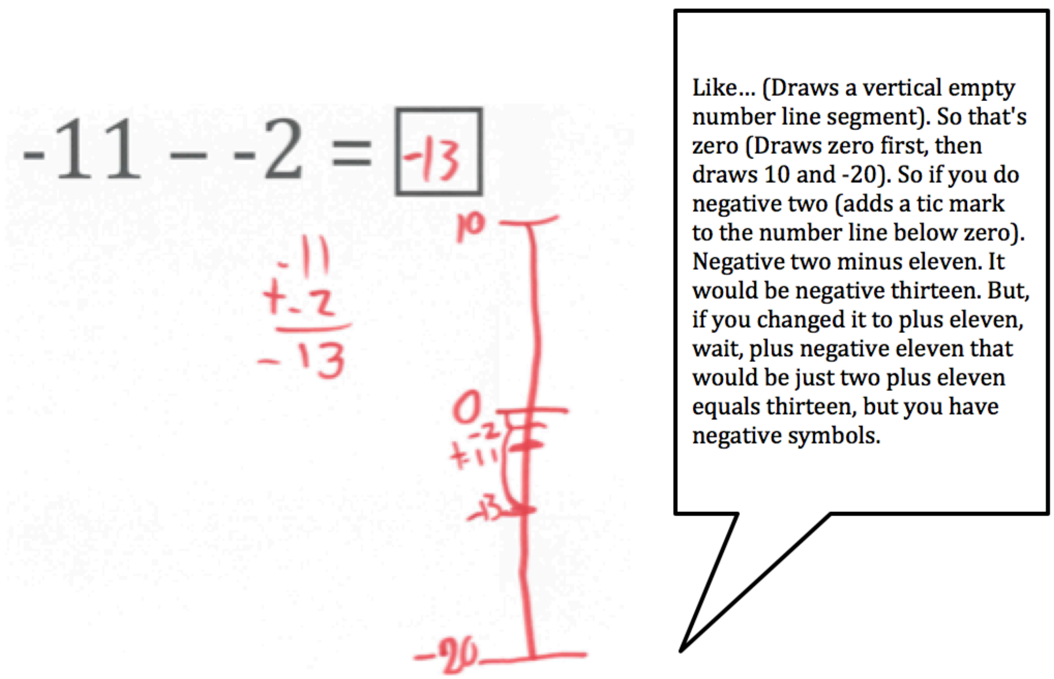


Figure 10. Jace drew a vertical Empty Number Line when he solved -11 – -2 = ☐, with an incorrect solution.

The children often used both the Number Line and Empty Number Line by starting with a particular number and then moving to another position. For instance, Figure 11 highlights how Kim used the Empty Number Lines when solving open number sentences in the last session. She subtracted a negative number by beginning with a number on the Empty Number Line and moving right to another position. As Kim solved ☐ – -2 = 1, she first thought the answer was 3 and then changed her solution and drew a Number Line (see Figure 11). She started at -1 and moved to the right to 0, "losing -1," and then moved right, losing another -1 to 1. She described this:

> Ok. (Writes 3 in box.) Wait. Yeah. I think so. The answer's a positive number ... but, wait, what? Wrong. (Whispers) Wait. It's not three (crosses of three with marker). I think I have another answer. Let me draw another box (draws a new box below the original box). I think it could be ... Negative one. The answer was one and here was a negative two (points at -2). So I sort of knew the only way I could get to a positive, which was the one (points at 1), which was to like have a smaller negative number (points at -2) besides 0 and then negative two. And, the only number was negative one (points at box with -1). And, if you did it, it was like a couple back in when I had the negative -9. It's pretty much just like that. So, the negative one, the two (starts drawing an Empty Number Line). This will be the biggest (marks the Empty Number Line with 1 and then 0). I don't know why I didn't make this smaller they are close numbers. (laughs) I will go with. I will go with negative two I think (draws -2 on the number line) and that should work. No (crosses off -2 on Empty Number Line). One (marks -1 on the Empty Number Line). When you subtract the two off of it, it would go, but when it hit zero it's lost one (marks Empty Number Line). So, it has zero. It has one remaining over, so you could just add onto and go into the positive area. And it, when you got done using your remainders it'd be one... I was thinking that it was a plus sign

right there (points at the subtraction symbol). So this (points at the subtraction symbol) it would be it's job would be to get to the positives.

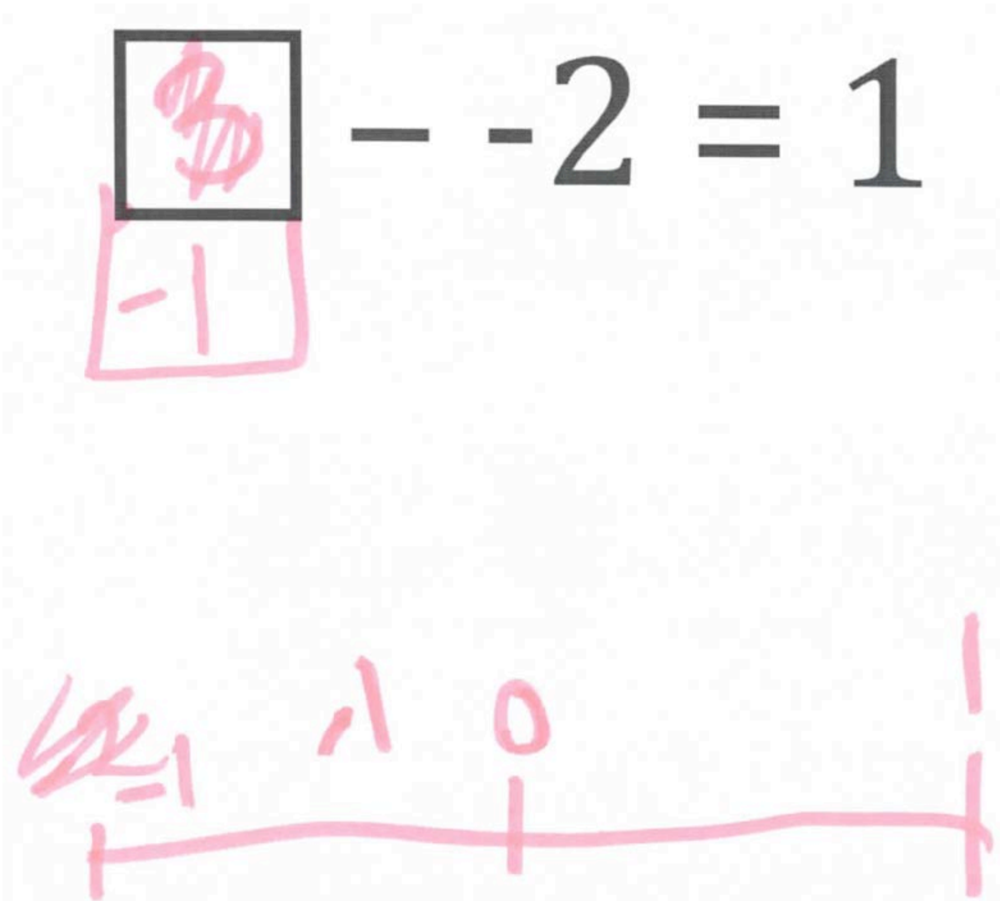


Figure 11. Kim drew a horizontal Empty Number Line when she solved ☐ – -2 = 1, with a correct solution.

Figures 11 illustrates how Kim used Empty Number Lines with integer subtraction, where she started at a point, -1, and moved right on the Empty Number Line to 1. Kim changed the traditional movement of subtraction and incorporated "losing" a negative distance.

5.5 Number Line

The children produced drawings that included Number Lines. The Number Lines included at least a segment of a number line that attempted equipartitioning. Number

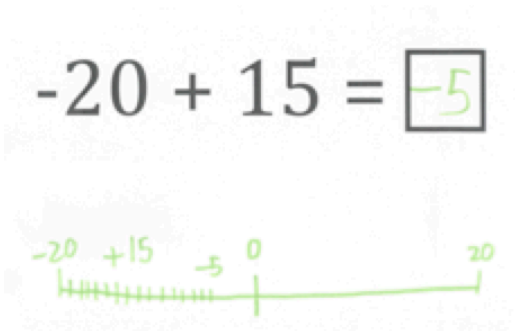
Lines did not necessarily have arrows indicating the infinitude in both directions, nor did Number Lines always have the entire segment equipartitioned.

Jace and Kim drew Number Lines infrequently and Alice did not draw any Number Lines. Jace drew 2 Numbers Lines out of 93 units (2%) and Kim drew 1 Number Line out of 93 units (1%). Overall, the children drew 3 Number Lines out of 279 units (1%).

Jace, for example, created a line segment from -20 to 20 when he first solved -20 + 15 = ☐ (see Figure 12). This was the first problem he solved in the first session and he began with equipartitioned units from -20 to -5 by starting with -20 and moving to the right 15 units drawing a tick mark each time. In later sessions, Jace used Empty Number Lines more.

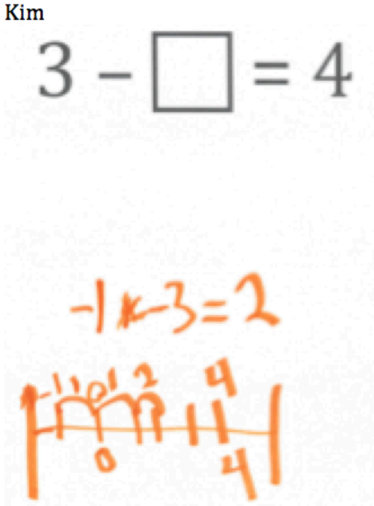
Similarly, Kim drew a line segment and partitioned her segment, from -2 to 5, as she worked on 3 – ☐ = 4 (see Figure 12). For Kim, she used this Number Line drawing as she attempted a problem in the second session. Although she did not determine a solution for solving this problem in the second session, her approach here is similar with the Empty Number Lines, used in later sessions and with eventual correct solutions.

Jace



Alright. Sometimes I like to draw number lines. (Starts drawing a number line.) This is my first time doing it with negative numbers. So, I'm going to see how it goes. So that's zero. So that would be right there. Let's just say this is ten. No, twenty. Because it's twenty back here out in the negatives. (Points to left end of the number line). And this is twenty; I mean, negative twenty (points to the left). So if you add fifteen, it would be fifteen going this way (motions to the right). One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen. (Draws tic marks). So that'd be plus fifteen (draws "+15" above number line) equals negative five). So, negative five. (Writes -5 in box.)

Kim



You start with a negative (draws on paper). But then the (continues drawing). That sign doesn't count (crosses off a part of her number sentence). That doesn't exist. Then it would equal just two because (finishes drawing a number sentence, -1 - 3 = 2 and starts drawing a number line). This would start off with negative one (indicates -1 on the number line) and there's zero (points at zero). This would be four to end with (marks four). So if you add three onto negative one, it would be hop one. That's already one wasted, one point wasted, you are at zero. Hop two more, but you end two short away from four.

Figure 12. Jace drew a horizontal Number Line when he solved -20 + 15 = ☐, with a correct solution in the first session. Kim drew a horizontal Number Line as she worked on 3 – ☐ = 4, with no solution determined, in the second session.

Similar to the Empty Number Lines drawings, the children did not produce Number lines always horizontally. Jace, for instance, drew a vertical Number Line when he solved 17 + ☐ = 8 (see Figure 13).

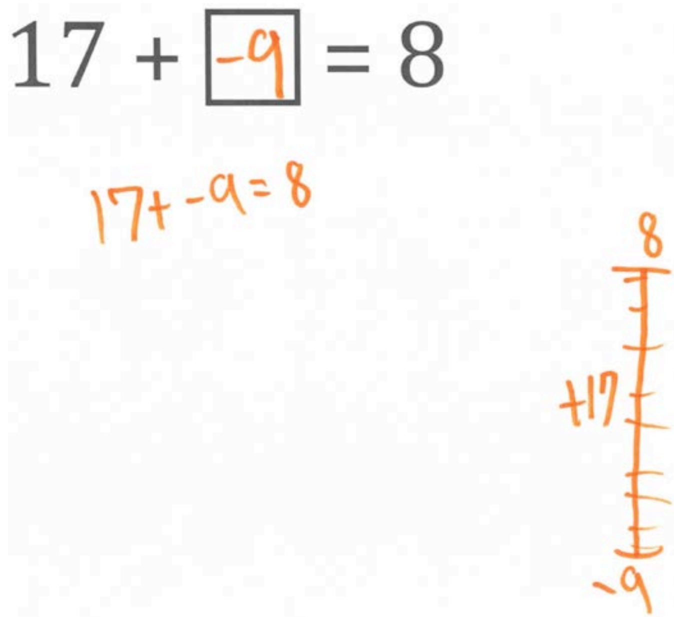


Figure 13. Jace drew a vertical Number Line when he solved 17 + ☐ = 8, with a correct solution.

As Jace produced this vertical Number Line, he used the Number Line as he explained his solution:

> (Draws horizontal problem. Writes 17 + -9 = 8). I think it's negative nine. Because it's, if you have negative nine and you add seventeen (Draws a line with -9 and 8 marked), you get eight (moves marker along his Number Line). The numbers in between... Let's just use tallies here, you know (draws tallies). I'm not counting them. But, the total tallies in between negative nine and eight is seventeen (draws "+17). And, that's how I'm ... you can check your answer.

Although Jace determined the solution -9, he did not actually use the Number Line because he drew an inaccurate amount of tallies on the Number Line. He also stated that there were 17 tallies, notably not 17 spaces of distance, between -9 and 8.

5.6 Vertical & Horizontal Number Sentences

The children produced drawings that included Horizontal or Vertical Number Sentences. The number sentences produced included re-expressions of the original number sentence written horizontally or vertically. Or, the number sentences produced included different number sentences that helped the children solve the open number sentence.

The children also produced both Horizontal and Vertical Number Sentences alongside other types of drawings, like the Number Line (see, e.g., Figure 13). Overall, the children produced Vertical Number Sentences in 87 out of 279 units (31%) and Horizontal Number Sentences in 47 out of 279 units (17%). Alice produced Vertical Number Sentences in 36 of 93 units (39%) and Horizontal Number Sentences in 6 of 93 units (6%). Jace produced Vertical Number Sentences in 51 of 93 units (55%) and Horizontal Number Sentences in 27 of 93 units (29%). Kim produced Vertical Number Sentences in 10 of 93 units (11%) and Horizontal Number Sentences in 17 of 93 units (18%).

All three of the children often drew Horizontal or Vertical Number Sentences as they solved the open number sentences. Sometimes these Horizontal or Vertical number sentences used only positive integers, while other times the horizontal or vertical number sentences incorporated negative integers. Figure 14 shows an example of some of the

Vertical Number Sentences that Alice drew as she solved -11 – -2 = ☐ and Horizontal

Number Sentences produced by Jace when he solved -5 – 4 = ☐.

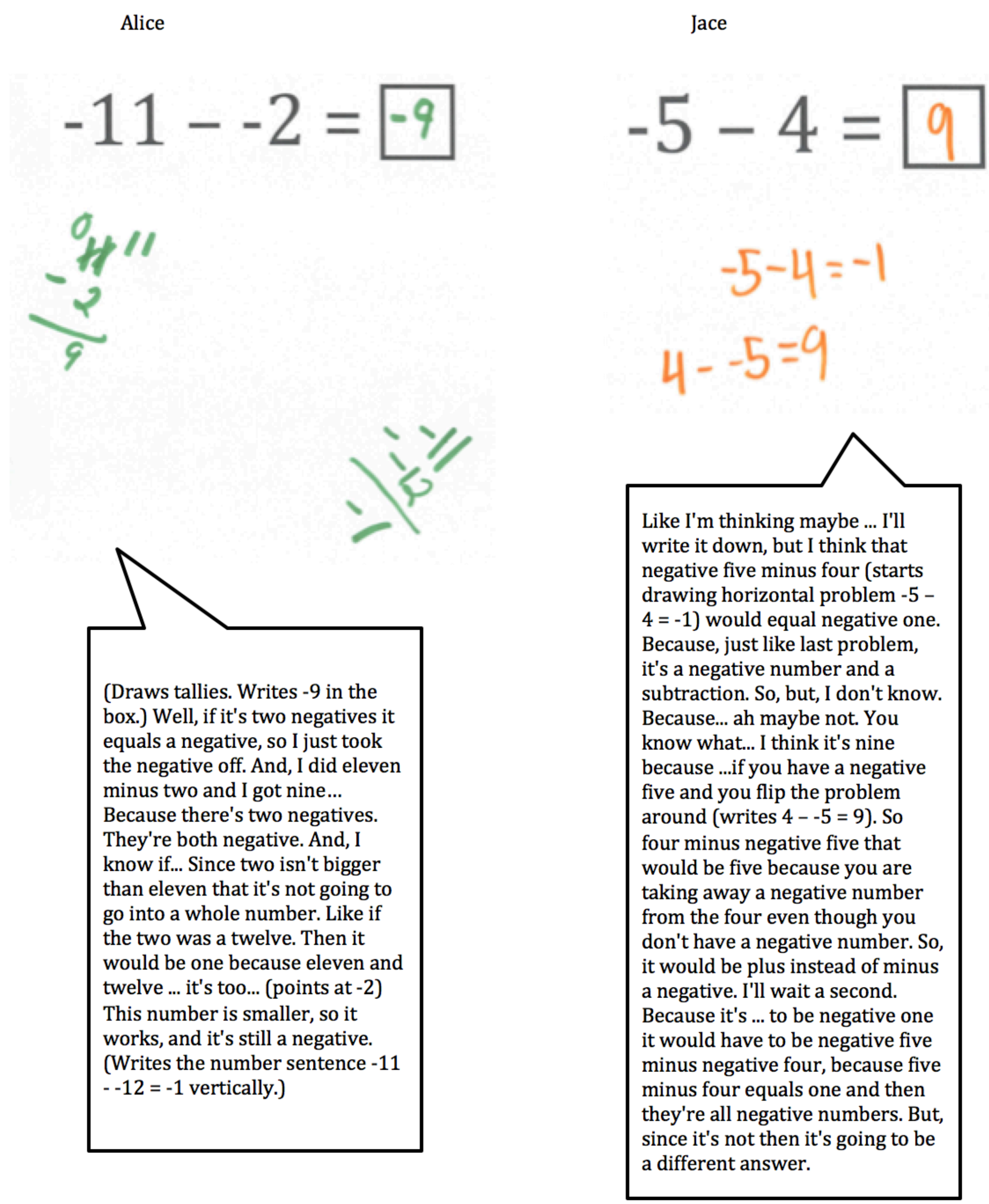


Figure 14. Vertical Number Sentences produced by Alice as she solved -11 – 2 = ☐, with a correct solution. Horizontal Number Sentences produced by Jace when he solved -5 – 4

= ☐, with an incorrect solution.

Alice first computed 11 – 2 as she solved -11 – -2 = ☐. As demonstrated in Figure 15, Alice drew a Vertical Number Sentence and applied a subtraction algorithm to 11 – 2 vertically. Alice also wrote a Vertical Number Sentence involving negative integers. Alice wrote a vertical number sentence equivalent to -11 – -12; however, she still obtained -1 as a solution. This is consistent to findings that children often apply the commutative property when subtracting negative integers (Bofferding, 2010). Also in Figure 14, Jace produced two different horizontal number sentences -5 – 4 = -1 and then 4 – -5 = 9. As Jace first solved the problem, he wrote -5 – 4 = -1—this was routine for Jace that he often gave for that problem type. However, he changed his original solution (-5 – 4 = -1) after he produced 4 – -5 = 9. Jace reasoned that if 4 – -5 = 9, then -5 – 4 would equal 9 as well, also consistent with Bofferding's (2010) findings about using the commutative property with subtraction.

5.7 Blank or Answer in Box Only

The children produced drawings that included no drawing and only the answer in the box. That is, the children used only verbal reasoning as they solved the open number sentence and wrote only the answer in the box. Or, the children produced no drawing because the problem was not solved.

The children produced 73 drawings out of 279 units (26%) that included an Answer in Box Only. And, the children produced 5 drawings out of 279 that were Blank (2%). Kim produced an Answer in the Box the most of the all of the children—in 57 of her 93 units (59%). Figure 15 highlights the Kim's Answer in Box Only, which she produced more than the other children.

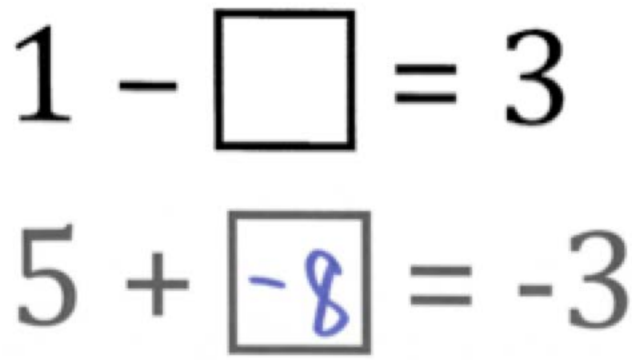


Figure 15. Blank drawing produced by Kim as she worked on 1 – ☐ = 3, with no solution determined and Answer in Box Only produced by Kim as she solved 5 + ☐ = 3, with a correct solution.

5.8 Plus, Minus, or Negative Sign Emphasis

The children produced drawings that included Plus, Minus, or Negative Sign Emphasis. These drawings included a plus sign, minus sign, or negative sign crossed off, circled, re-written, or highlighted in some way.

The children made a Plus, Minus, or Negative Sign Emphasis in 20 of the total 279 units (7%). Jace produced these drawings in 13 of his 93 units (14%). Kim made these in 1of her 93 units and Alice made these in 6 of her 93 units (6%).

Figure 16 highlights how Jace crossed off a plus sign. He produced an emphasis on the sign when stated that, “you take away the addition symbol” and crossed off the plus sign.

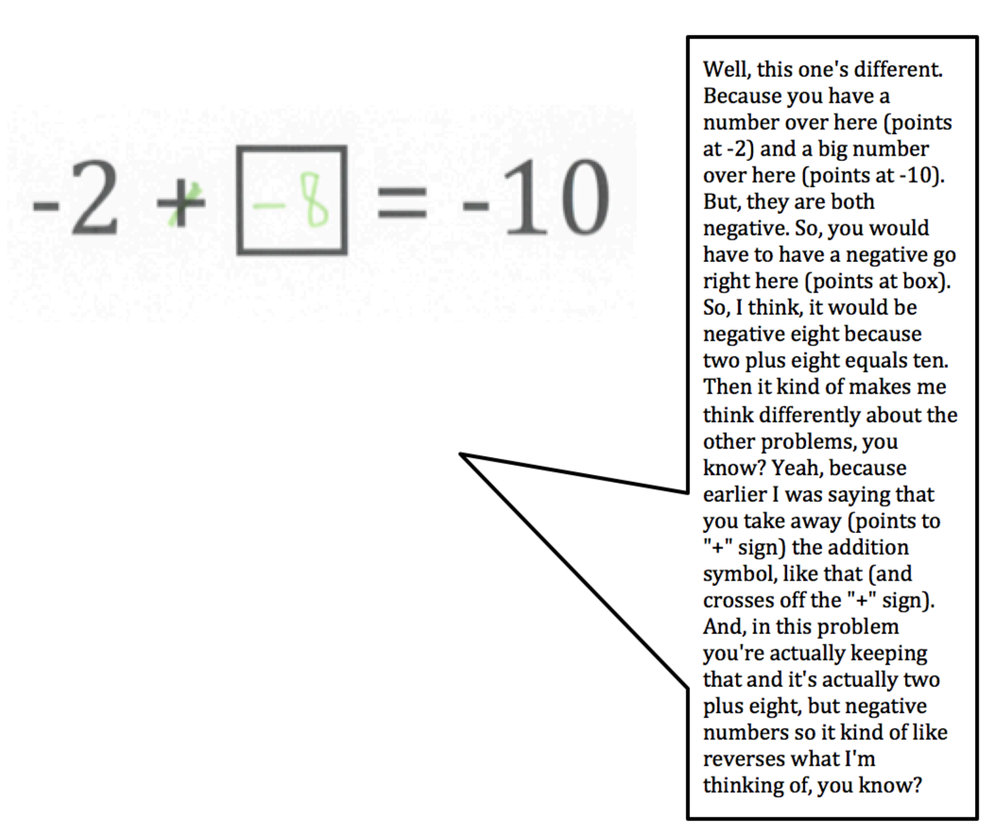


Figure 16. Plus sign emphasis produced by Jace as he solved -2 + ☐ = -10, with a correct solution.

Figure 17 illustrates how Jace crossed off a minus sign and circled the negative sign ☐ – -1 = 4 and he crossed the plus sign off and wrote a Vertical Number Sentence 19 – -2 = -17. Also, in Figure 17, when Jace crossed off the minus sign he produced Vertical Number Sentences also. As Jace produced the drawing in Figure 17, he shared that the minus sign could be crossed off in the number sentence ☐ – -1 = 4 because “you can ignore the plus sign when you solve box plus -1 equals 4 (☐ + -1 = 4).” When Jace solved ☐ + -2 = 17, He crossed off the plus sign, stating that it can be ignored and interpreting the adding of negative two as subtraction of positive two. However, he wrote 19 – -2 rather than what he verbally computed, 19 – 2.

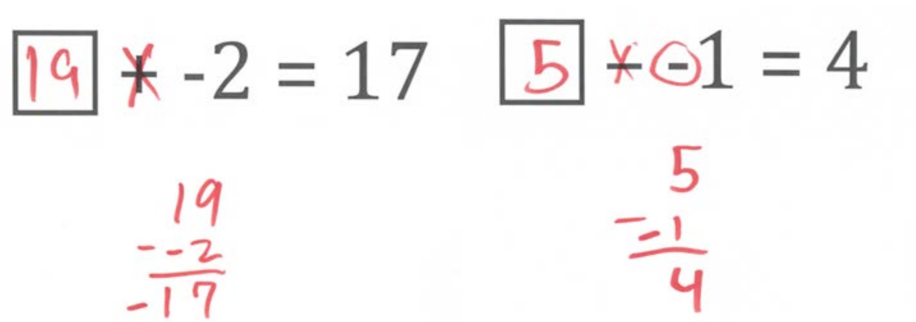


Figure 17. Plus sign emphasis produced by Jace as he solved ☐ + -2 = 17, with a correct solution. Minus sign and negative sign emphasis produced by Jace as he solved ☐ – -1 = 4, with an incorrect solution.

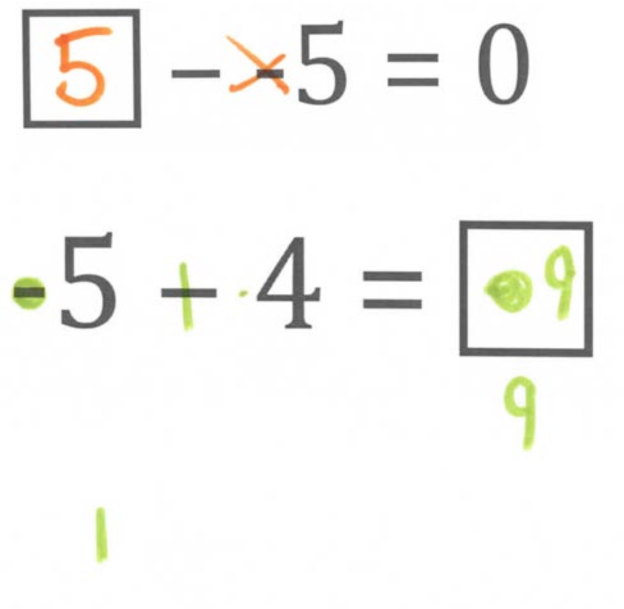


Figure 18. Kim solved ☐ – -5 = 0 and produced a Negative Sign Emphasis, with an incorrect solution. Alice solved -5 – 4 = ☐ and produced a Negative and Minus Sign Emphasis, with an incorrect solution.

Figure 18 illustrates that Kim and Alice also placed Negative and Minus Sign Emphasis in ways that did not produce the correct solutions. When emphasis was placed on plus signs, it produced correct answers (see Figure 17), yet when emphasis focused on negative or minus signs it did not result in correct answers (see Figures 17 and 18).

6. Use and Changes in Learner-Generated Integer Drawings

Table 2 highlights the different drawings used with both the counts and percentages for the children individually and overall.

Table 2

Different Drawings and Use (Counts & Percentages) by Grade 5 Children

	Alice (*n* = 93)	Kim (*n* = 93)	Jace (*n* = 93)	Overall Use (*n* = 279)
Single Set of Objects	20 (22%)	0 (0%)	1 (1%)	21 (8%)
Double Set of Objects	38 (41%)	0 (0%)	0 (0%)	38 (14%)
Number Sequence	2 (2%)	0 (0%)	4 (4%)	6 (2%)
Empty Number Line	0 (0%)	5 (5%)	6 (6%)	11 (4%)
Number Line	0 (0%)	1 (1%)	2 (2%)	3 (1%)
Horizontal Number Sentence	6 (6%)	17 (18%)	27 (29%)	47 (17%)
Vertical Number Sentence	36 (39%)	10 (11%)	51 (55%)	97 (35%)
Blank	0 (0%)	3 (3%)	2 (2%)	5 (2%)
Answer in Box Only	8 (9%)	56 (60%)	9 (10%)	73 (26%)
Plus, Minus, or Negative Sign Emphasis	6 (6%)	1 (1%)	13 (14%)	20 (7%)

6.1 Changes in drawing choice across sessions

Alice, Jace, and Kim produced different drawings across the Individual Open Number Sentence Sessions (see Tables 3, 4, and 5, respectively). As a strategy, these variations in drawing productions paired with correct and incorrect solutions differently. Changes in the children's drawing productions will be described next.

6.2 Alice's drawing use & changes

Table 3

Drawings Produced by Alice Across Sessions

	Individual Session 1 (*n* = 20)	Individual Session 2 (*n* = 23)	Individual Session 3 (*n* = 25)	Individual Session 4 (*n* = 25)
Single Set of Objects	4 (20%) 3 Correct 1 Incorrect	7 (30%) 2 Correct 5 Incorrect	7 (28%) 5 Correct 2 Incorrect	5 (12%) 4 Correct 1 Incorrect
Double Set of Objects	2 (10%) All Correct	10 (43%) 5 Correct 5 Incorrect	13 (52%) 8 Correct 5 Incorrect	13 (52%) 8 Correct 5 Incorrect
Number Sequence	0 (0%)	0 (0%)	0 (0%)	2 (8%) All Correct
Empty Number Line	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Number Line	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Horizontal Number Sentence	0 (0%)	1 (4%) Correct	1 (4%) Correct	4 (16%) 1 Correct 3 Incorrect
Vertical Number Sentence	15 (75%) 8 Correct 7 Incorrect	10 (43%) 8 Correct 2 Incorrect	7 (28%) 5 Correct 2 Incorrect	4 (16%) 3 Correct 1 Incorrect
Blank	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Answer in Box Only	0 (0%)	0 (0%)	1 (4%) Incorrect	7 (28%) All Correct

Plus, Minus, or Negative Sign Emphasis	0 (0%)	0 (0%)	0 (0%)	6 (24%) 2 Correct 4 Incorrect

Alice produced Single Set of Objects, Double Set of Objects, and Vertical Number Sentences in all four sessions (see Table 3). Her use of Single Set of Objects seemed consistent across the sessions and use of Double Set of Objects increased in the second session. Her production of Vertical Number Sentences decreased from the first individual session to the last. Alice did not produce a Number Sequence or Plus, Minus, or Negative Sign Emphasis until the last session. She produced no Blank, Empty Number Line, or Number Line drawings.

The Double Set of Objects drawings Alice produced paired with both correct and incorrect solutions, but the incorrect solutions decreased over time. Similarly, as Alice created Single Set of Objects, these drawings paired with correct answers more than half of the time. Other drawings like, Vertical Number Sentences and Answer in Box Only, typically paired with correct solutions, and Plus, Minus, or Negative Sign Emphasis with incorrect solutions. She used the Number Sequence, although paired with correct solutions, with other drawings, Double Sets of Objects and Vertical Number Sentences.

6.3 Jace's drawing use & changes

Table 4

Drawings Produced by Jace Across Sessions

	Individual Session 1 (*n* = 20)	Individual Session 2 (*n* = 23)	Individual Session 3 (*n* = 25)	Individual Session 4 (*n* = 25)
Single Set of Objects	1 (5%) Correct	0 (0%)	0 (0%)	0 (0%)

Double Set of Objects	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Number Sequence	1 (5%) Correct	0 (0%)	1 (4%) Incorrect	2 (8%) All Correct
Empty Number Line	3 (15%) All Correct	3 (13%) All Correct	0 (0%)	0 (0%)
Number Line	1 (5%) Correct	0 (0%)	0 (0%)	1 (4%) Correct
Horizontal Number Sentence	0 (0%)	3 (13%) 3 Correct	0 (0%)	24 (96%) 22 Correct 2 Incorrect
Vertical Number Sentence	7 (35%) 4 Correct 3 Incorrect	19 (83%) 11 Correct 8 Incorrect	23 (92%) 17 Correct 6 Incorrect	2 (8%) 2 Correct
Blank	0 (0%)	1 (4%) Incorrect	1 (5%) Incorrect	0 (0%)
Answer in Box Only	9 (45%) 4 Correct 5 Incorrect	0 (0%)	0 (0%)	0 (0%)
Plus, Minus, or Negative Sign Emphasis	3 (15%) All Correct	7 (30%) 1 Correct 6 Incorrect	1 (4%) Correct	2 (8%) 2 Correct

In the first session, Jace mostly produced Answers in Box Only, but also created Single Set of Objects, Number Sequence, Empty Number, Number Line, and Plus, Minus, or Negative Sign Emphasis drawings. After the first session, where almost half of the time he produced Answer in Box Only, he created other drawings in the other sessions (see Table 4). Although Jace produced a Single Set of Objects in the first individual session, he did not produce that drawing in the remaining sessions. Jace never produced a Double Set of Objects drawing. He created Empty Number Lines in the first two sessions and did not produce any more for solving open number sentences. He produced Number Lines scarcely, once in the first session and once in the last session. Jace's use of Empty Number Line and Number Line decreased across the sessions. Jace made Plus, Minus, or Negative Sign Emphasis across the sessions, with the most frequent use in the second session.

In the first session, when Jace produced a Single Set of Objects, a Number Sequence, Empty Number Lines, a Number Line, or Sign Emphasis, he obtained correct solutions. Although he also obtained correct solutions with Vertical Number Sentences and Answers in Box Only, these are the only drawings he produced with incorrect solutions. By the second session, he produced an Emphasis on Signs paired mostly with incorrect solutions and he used Emphasis on Signs less in the third and fourth sessions. His use of Empty Number Line and Horizontal Number Sentence drawings paired with all correct solutions in the second individual session. Also in the second individual session, his use of Vertical Number Line drawings paired with both correct and incorrect answers. By the last session, Jace obtained nearly all correct answers, with the exception of two problems, and wrote horizontal number sentences for all problems but one. Although Jace did not produce a variety of drawings in in last session, he still produced a Number Sequence and Number Line for more challenging problems types for him (i.e., -a – b, where a > b > 0).

6.4 Kim's drawing use & changes

Table 5

Drawings Produced by Kim Across Sessions

	Individual Session 1 (*n* = 20)	Individual Session 2 (*n* = 23)	Individual Session 3 (*n* = 25)	Individual Session 4 (*n* = 25)
Single Set of Objects	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Double Set of Objects	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Number Sequence	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Empty Number Line	0 (0%)	0 (0%)	0 (0%)	5 (20%) All Correct
Number Line	0 (0%)	0 (0%)	1 (4%) Incorrect	0 (0%)
Horizontal Number Sentence	0 (0%)	1 (4%) Correct	13 (52%)	3 (12%) 2 Correct 1 Incorrect
Vertical Number Sentence	0 (0%)	5 (22%) All Correct	4 (16%) All Correct	1 (4%) Correct
Blank	0 (0%)	1 (4%) Incorrect	0 (0%)	2 (8%) Incorrect
Answer in Box Only	20 (100%) 10 Correct 10 Incorrect	16 (70%) 14 Correct 2 Incorrect	6 (24%) 4 Correct 2 Incorrect	14 (56%) 12 Correct 2 Incorrect
Plus, Minus, or Negative Sign Emphasis	0 (0%)	0 (0%)	1 (4%) Incorrect	0 (0%)

Kim produced Answer in Box Only drawings in the first session for all of the number sentences she solved. Beginning in the second individual session, Kim produced other drawings, like Horizontal and Vertical Number Sentences (see Table 5). Beginning in the third session, Kim produced other drawings, such as Number Line. Kim did not produce a Single Set of Objects or a Double Set of Objects in any of the four individual sessions. In the last session Kim drew Empty Number Lines. Kim started drawing both Horizontal and Vertical Number sentences in the second sessions. The use of the Vertical Number Sentence decreased through the rest of the session and the use of the Horizontal Number Sentences increased and then decreased. Kim sporadically, with little frequency, produced Number Line, Blank, and Plus, Minus, or Negative Sign Emphasis drawings. Kim’s most produced drawing, Answer in Box Only, paired with correct answers more over time. When Kim produced Vertical Number Sentence drawings, these paired with correct solutions all of the time. Although Kim produced a Number Line, with an incorrect solution in the third session, in the last session she created several Empty Numbers Lines, paired with correct solutions.

7. Discussion

Learner-generated integer drawings are highlighted in this article by the drawings produced by three Grade 5 children: Single Set of Objects, Double Set of Objects, Number Sequence, Number Line, Empty Number Line, Horizontal Number Sentence, Vertical Number Sentence, Plus, Minus, or Negative Sign Emphasis, Answer in Box Only, and Blank. Also, strategic learner-generated integer drawings are illustrated with descriptions of how they solved the problems. Similarly, an examination into the frequency of how Alice, Jace, and Kim used these drawings differently over time is provided. The significance of types of drawings created, and changes in how the children used them, will be discussed next.

7.1 Significance of the study

The significance of this study is highlighted in Table 6.

Table 6

Significance of Study

The results of this study:
(1) … offer a new framework that highlights children’s representations of integer addition and subtraction. Yet, this framework of drawings for integer addition and subtraction is compatible with the various descriptions of student thinking in the literature, providing further insight into children’s thinking and potentially connecting existing frameworks.
(2) … illustrate nuances of the various drawings, which are necessary for leveraging student constructions within instruction about integer addition and subtraction. Describing the uniqueness of the children’s drawings distinguishes these drawings from existing instructional models.
(3) ... demonstrate changes in drawings over time, illustrating a potential developmental perspective for integer drawings. Although previous research has highlighted cross-sectional perspectives on thinking about integers, these descriptions of changes uniquely provide insight into what learning may occur as children engage with integers.

7.1.1 A new framework of drawings for integer addition and subtraction

Although there are descriptions about the ways that children may reason about integers (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014), different mental models that children may have for integers (Bofferding, 2014), sophisticated reasoning that children may employ (e.g., Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Featherstone, 2000), and various instructional models for integer operations (e.g., Battista, 1983; Janvier, 1985; Saxe et al., 2013), the results of this study provide a perspective on student thinking about integers by describing the drawings produced by children (Sfard, 2008; van Meter & Garner, 2005).

Although drawings, like Number Line (e.g., Bishop et al., 2011), are evident in prior literature, this framework highlights unexpected nuances of learner-generated drawings. Specifically, the children represented negative integers on the right side of a horizontal Empty Number Line or on top of a vertical Number Line. This framework also provides new insight into drawings, such as the creations of Single and Double Sets of Objects, not illustrated in prior literature.

The framework of learner-generated drawings for integer addition and subtraction complements other frameworks, such as mental models or ways of reasoning, as it is explicitly connected to children's thinking. The Double Set of Objects, for instance, may be related to the counterbalance conceptual model as it deals with two sets of quantities (Wessman-Enzinger & Mooney, 2014; Wessman-Enzinger, 2015); yet, Double Set of Objects is different from cancellations models (e.g., Battista, 1983; Koukkoufis & Williams, 2006; Linchevski & Williams, 1999; Vig et al., 2014) as often the children compared two positive quantities (e.g., $6 - 8 = \square$). Along these lines, the Double Set of Objects may provide insight into analogy-based ways of reasoning by Bishop et al. (2016) as magnitude comparisons are included in their definition. In terms of the Empty Number Lines and Number Lines, this is likely related to order-based reasoning (Bishop et al., 2016) and translation or relativity conceptual models (Wessman-Enzinger & Mooney, 2014; Wessman-Enzinger, 2015). The use of flexible orderings (e.g., constructions with negatives on the right) may provide perspective into different mental models used (Bofferding, 2014).

The framework of learner-generated drawings for integer addition and subtraction illustrates what the children constructed as they solved integer open number sentences (van Meter & Garner, 2005) and provides perspective into their thinking about integers (Sfard, 2008). Children's thinking is often different than adults (Carpenter et al., 2015; Bofferding, 2010; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014) and this framework represents children's unique perspective of integer drawings. Because this study was conducted prior to school instruction about integers and the children were not provided manipulatives or explicit instructional models throughout the entire study, their drawing productions highlight both the sophisticated nature of their mathematical inventions and the challenges they will have as they learn about integer addition and subtraction.

7.1.2 Re-evaluating instructional models

Although we have insight into the types of drawings that children may produce for whole numbers (e.g., Carpenter et al., 2015), the presented framework illustrates what learner-generated drawings look like for children that engage with integers. The types of drawings produced in this study are not the same as for whole numbers. Consequently, the children produced different types of drawings with objects for integers rather than whole numbers. Adding or removing changes in objects to singular sets of objects (Single Set of Objects) or comparing two sets of objects (Double Set of Objects) represents one of these nuanced difference in integer drawings. A challenge that educators may face in using these drawings is that although the Double Set of Objects may align to many integer instructional models (e.g., cancellation models, use of two-colored chips, contexts like electron charges), the Single Set of Objects does not. For this reason, the distinction of Single Set of Objects drawings in this framework supports understanding student thinking and incorporating children's thinking into integer instructional models.

Drawings that include a Single Set of Objects do not align with integer instructional models currently advocated.

Although we know that students may use a number line provided (e.g., Saxe et al., 2013) or draw number lines (e.g., Bishop et al., 2011), we needed collective insight about the types of drawings produced that may support number lines. One such nuance in those types of drawings is the distinction between a Number Sequence from a Number Path (Bofferding, 2010), not highlighted in prior literature. Another nuance highlighted includes the use of negative integers on Empty Number Lines and Numbers Lines in unconventional places. Specifically, the children used negatives on the right side of a horizontal Number Line or on the top of a vertical Number Line—differing from cultural conventions and contemporary instructional models.

The Grade 5 children used certain drawings with different frequencies. For example, Alice used Single and Double Objects the most, while the other children did not create this type of drawing frequently. Yet, with the exception of the Double Set of Objects, at least two of the children used all of the different types drawings at some point—highlighting that other children may be capable of producing these drawings given opportunities for creating and developing their own uses of integers. The children produced some drawings more than others—illustrating that children do not think about the integers the same. Thus, top-down approaches where students use a specified or required instructional model (e.g., a cancellation model, a number line model), rather than create their own constructive and representative drawings, may not be the best way for beginning integer instruction. Rather, children should produce and construct their own drawings that represent the integers, which could potentially serve as the instructional models.

Yet, at least two of the children produced most of these drawings at some point. This highlights consistency in the ways that the children may think about integers and the drawings that they produce. Thus, this framework of integer drawings points to common drawings that can be supported in instruction. Because instructional models all have breaking points (Vig et al., 2014), this framework paired with the incorrect and correct solutions provides an inaugural space for understanding the affordances and limitations of children's drawings, which could potentially serve as their instructional models.

7.1.3 Changes in learner-generated integer drawings

The children in this study also produced drawings that changed over time. Kim, for example, did not produce any drawings in the first session and began producing more drawings in subsequent sessions—illustrating that drawing production and development takes time. Alice consistently produced Single and Double Objects and produced her first Number Sequence in the last session. Kim also produced Empty Number Lines for the first time in the last session. Both of these examples of drawings that support order (e.g., Number Sequence, Empty Number Line) highlight that development of the Number Line may take time.

Their unique drawing use and changes in those drawings highlight potential developmental perspectives of drawings, like constructing Number Lines. In relationship to current integer instruction, these results highlight that children develop and create models differently—illustrating potential learning progressions for drawing use. Although other research has pointed to cross-sectional differences with integers (e.g., Whitacre, 2017), an important element of microgenetic design (Seigler & Crowley, 1991) is that learning be investigated with the same individuals over a span of time for authentically understanding development.

Furthermore, if the development of Number Line drawings takes time and children produce novel constructions of the Number Line, this questions the authenticity of "top-down" approaches of giving students instructional models for integer addition and subtraction. Rather, we should be investigating "bottom-up" approaches with integer instructional approaches. Instead of giving students an instructional model and investigating their thinking after use of this model, we need investigations that support students in *their* learning, not our vision of learning, even for such things as the drawings that they produce.

8. Conclusion

The results of this study are significant as they include a classification of the types of drawings that children created and provide insight into development, or changes over time, in these drawing productions. Within this classification, the children often created drawings that incorporated discrete objects (e.g., Single Set of Objects, Double Set of Objects) as well as continuous objects (e.g., Empty Number Lines, Number Lines). More investigations are needed that explore the ways that more children produce these drawings and how the production of these drawings develops over time. Because the children drew the continuous objects less than other drawings, perhaps the production and creation of continuous objects develops from the production of discrete objects, taking longer time than traditional integer instructional models provide. Similarly, the production of discrete objects in the drawings may also be developmental, decreasing in production as the children work more with integer operations. Deeper investigations into the learning and development, beyond a classification for drawings for integer addition and subtraction is needed. Overall, these results call for a re-examination of current integer instructional models and support advocating instructional approaches based on children's constructed drawings. These different types of drawings may help leverage future research on student thinking about integer operations.

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