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Integer Play and Playing with Integers (Chapter Two of Exploring the Integer Addition and Subtraction Landscape: Perspectives on Integer Thinking)

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Abstract This chapter describes instances of play within a teaching episode on integer addition and subtraction. Specifically, this chapter makes the theoretical distinction between integer play and playing with integers. Describing instances of integer play and playing with integers is important for facilitating this type of intellectual play in the future. The playful curiosities arising out of integer addition and subtraction tended to be concepts that we think of prerequisite knowledge (e.g., magnitude or order, sign of zero) or knowledge that is more nuanced for integer addition and subtraction (e.g., how negative and positive integers can "balance" each other). Instances of integer play and playing with integers are connected to the work of mathematicians, highlighting the importance of play in school mathematics.

Embracing the identity of a mathematician or participating in the work of a mathematician may seem like a foreign idea, especially to elementary school students. Yet, children are more capable of approaching mathematics similar to research mathematicians than they realize:

> Young children develop mathematical strategies, grapple with important mathematical ideas, use mathematics in their play, and play with mathematics. Young children often enjoy their mathematical work and play. Indeed, despite its immaturity, young children's mathematics bears some resemblance to research mathematicians' activity. Both young children and mathematicians ask and think about deep questions, invent solutions, apply mathematics to solve real problems, and play with mathematics. (Ginsburg, 2006, p. 158)

A key idea expressed by Ginsburg is the idea of play. He posits that through play students deeply engage in mathematics, reminiscent of mathematicians. The idea of fusing play with mathematics comes at a pivotal time in education and society. Increased educational testing (Ravitch, 2010), demands to meet expectations of standards (e.g., National Governors Association Center for Best Practices and

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Council of Chief State School Officers [NGA and CCSSO], 2010), and increased needs for children to pursue STEM careers in the future (Ellis, Fosdick, & Rasmussen, 2016; Olson & Riordan, 2012) are just some of the contemporary pressures. As stress continues to build around the increase in testing and expectations in standards, there is also a push to extend play throughout elementary school (Parks, 2015). Including play in mathematics may reduce stressful mathematical experiences. Engaging children in playful experiences of mathematicians may also have the potential to provide increased opportunities for access to more complex mathematical concepts. Although there are calls for mathematical play (Ginsburg, 2006) and prolonged play in school (Parks, 2015), most of these play experiences are described with young children. But, are children in late elementary school able to learn advanced mathematical concepts through play? This chapter illuminates the potential of play for supporting children's mathematical thinking and learning about integers and integer operations in Grade 5.

Elements of Play

Children, like research mathematicians, engage in mathematical play and playful mathematics (Ginsburg, 2006). Ginsburg classified mathematical play as engaging in mathematics embedded in play. For instance, when building block towers, children may count their blocks or compare the heights of block towers as they play. Ginsburg also classified playful mathematics as play centered on mathematics. This may happen when students engage in play that is purposefully mathematical—like playing a walking game on a number line.

These types of play, mathematical play and playful mathematics, should not be reserved for just young children (Parks, 2015) or just mathematical topics typically advocated at their grade level (Featherstone, 2000). Play can help them investigate new concepts as well. Parks (2015) lamented about the need for play throughout elementary school, "as children move through the primary grades and have fewer and fewer opportunities for play, finding ways to bring choice, excitement, movement, imagination, and curiosity into formal lessons becomes more and more important" (p. 112). We know that children are capable of sophisticated reasoning about integers (Bofferding, 2014) and integer addition and subtraction (Bishop et al., 2014). Mathematical play and playful mathematics may be a space for children to engage in topics, like integers, at later elementary grades and before age levels recommended in standards (NGA & CCSSO, 2010).

Identifying elements of mathematical play and playful mathematics (see Table 2.1), even with older children, can help distinguish the intellectual, but playful, experiences that children engage in as they play with integers (Featherstone, 2000; Parks, 2015). Burghardt (2011) described essential criteria for play: spontaneous or pleasurable, not fully functional, different from similar serious behaviors, repeated, and initiated in the absence of stress. First, play must be spontaneous and pleasurable—it is a necessary requirement that play is fun and enjoyable for children.

Table 2.1 Elements of play

Criteria for play (Burghardt, 2011)	Additional criteria for play (Parks, 2015)
Spontaneous or pleasurable	Opportunities for social engagement
Not fully functional	Creative thinking
Different from similar serious behaviors	Appealing materials
Repeated	Physical movement
Initiated in the absence of stress	Imagination

Second, play is not fully functional because it is not necessary for survival but has some functional aspect. Play may be functional, like building a castle out of blocks for a doll. In this way, play may serve some sort of function and have delayed benefits. Third, play must also include some qualities that differentiate it from serious behaviors. Children dancing or pretending to be an animal, for example, are different than typical behaviors in the surrounding environment. Fourth, play also includes elements of repetition because children will often repeatedly play until a skill is mastered. For example, children may try to build a tall block tower. As they build this tower, it may topple over, but they will continue to repeatedly build this tower until it stands. Last, play must be initiated in the absence of stress—play is voluntary and takes place in a safe, relaxed environment. Burghardt noted that all of these criteria must be met in some capacity for true play. However, play includes additional criteria, such as social engagement, creative thinking, appealing materials, physical movement, and imagination (Parks, 2015).

Insight into how these elements of play are present as children engage in integer play and play with integers is needed. Identifying elements of play and describing instances of them provides insight into the opportunities and spaces for deep, intellectual, and mathematical thought. Describing instances of intellectual play may also offer insight into how play may be supported in school mathematics throughout elementary school.

Imaginative Play Supports Thinking and Learning About Integers

One of the prevalent themes in the literature across history is that the thinking and learning about integer addition and subtraction is notoriously challenging (e.g., Bishop et al., 2014; Piaget, 1948; Thomaidis, 1993). Yet, we are gaining deeper insights into the ways that children think about integers (Bofferding, 2014) and integer addition and subtraction (Bishop et al., 2014; Bofferding, 2010; see Chap. 3). One reason the negative integers may be so challenging is the lack of physical embodiment of them (Martínez, 2006; Peled & Carraher, 2008). That is, the negative integers cannot be used as objects that physically exist (e.g., -2 fish) without opposites and an abstract one-to-one mapping of an integer to an object (e.g., stating that a red chip represents -1). Because of the physical constraints of the negative integers, the integers are not as naturally accessible in play as the whole numbers or natural numbers.

Even so, Featherstone (2000) illustrated that play can be built around the imaginative world of integers. She presented an illustration of a Grade 3 student journaling about additive inverses in a playful way: "-pat + pat = 0" (p. 14). Educators and researchers also utilize games for the teaching and learning of integer addition and subtraction (e.g., Bofferding & Hoffman, 2014; Wessman-Enzinger & Bofferding, 2014). Games may be the space to encourage the imaginative mathematical play that Featherstone discussed. Play often centers on mathematics (Ginsburg, 2006; Parks, 2015), such as playing on a linear board game (e.g., Bofferding & Hoffman, 2014; Siegler & Ramani, 2009).

Yet, mathematical play is not just playing with a game but is *play with numbers* (Featherstone, 2000; Ginsburg, 2006; Steffe & Wiegel, 1994). Boaler (2016) supports play with numbers for all students: "The best and most important start we can give our students is to encourage them to play with numbers and shapes, thinking about what patterns and ideas they can see" (p. 34). Featherstone (2000) argued that as children engage with integers, this may be a "territory for mathematically imaginative play" (p. 20). She also connected some features of play to exploring integers in elementary school. For example, one of the defined attributes of play is that play exists in a separate, outside world (Huizinga, 1955). That is, the child is able to step outside of reality into this other world. Featherstone (2000) proposed that the integers themselves are this imaginative world. She wrote, "The territory below zero is a separate world for elementary students. It is an outside the 'real' world of natural numbers - numbers that are in daily use both inside and outside of school" (p. 20). This type of imaginative play may be a way to share integers sooner and prolong play in schools.

We need more descriptions and insight into this imaginary world of play with integers that the children often step into. This chapter highlights how these different types of play, playing a game and engaging in mathematically imaginative play, work together to support thinking and learning about integers and integer addition and subtraction. Specially, this chapter illustrates specific instances of integer play and playing with integers and connects these instances to the elements of play described by Burghardt (2011) and Parks (2015). Then, these instances of play (integer play and playing with integers) are connected to the work of research mathematicians to show the potential for play in upper elementary grades.

Context of the Study

The data reported on in this chapter comes from a 12-week teaching experiment (Steffe & Thompson, 2000) with three Grade 5 students designed to examine the teaching and learning of integers, specifically negative integers. The teaching experiment was comprised of nine group sessions and eight individual sessions for each child. During these sessions, the students were introduced to four conceptual models for integer addition and subtraction ([CMIAS], Wessman-Enzinger & Mooney, 2014)—bookkeeping, counterbalance, translation, and relativity—through the use of various contextualized problems and activities. Although these CMIAS were introduced throughout the teaching experiment, it was not expected that the students would use only these models; there were opportunities for students to think about the addition and subtraction of integers freely as they engaged in activities during the group sessions.

The mathematics that the students discussed and the misconceptions they held influenced the content and development of the group sessions of the teaching experiment. I served as the teacher-researcher for this teaching experiment. A second researcher was the witness for most of the group sessions. He took field notes during the group sessions. In addition to taking field notes, he also periodically asked questions of the participants during the sessions. After each group session with the students, the witness and I debriefed about the students' thinking and learning that appeared to be emerging during the sessions. We also discussed plans for the next group session, and I considered his observations and suggestions for the next instructional moves, based on the students' responses in that session. After each individual and group session, I wrote reflections about what I noticed as the teacher-researcher, what I thought the next instructional moves should be, and why I thought that move should be made.

The focus of this chapter is on the fourth group session because it serves as an example of the playful mathematics imbedded in mathematical play (Ginsburg, 2006). This group session incorporated playing an integer-focused card game, during which the students engaged with mathematics in ways that we had not planned. I present this case to illustrate the power of mathematical play for creating opportunities for play with mathematics and to show how such play can support mathematical thinking.

Integer Play

The mathematical goals of the integer play in the group session constituted adding integers, subtracting integers, developing a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014), and distinguishing the minus symbol from the negative symbol.

In the card game, *Integers: Draw or Discard*, drawing cards aligned with integer addition and discarding cards aligned with integer subtraction (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014); therefore, the game fostered discussion of both addition and subtraction during this session. After including the drawn cards to their hand, the children determined their total points of their cards by adding. Discarding a card was similar to subtraction—as the point value of the card was taken away from the total hand. Thus, if students discarded a negative integer card, they considered the effects of subtracting a negative integer.

Developing thinking about integers with a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014) constituted another mathematical goal of the integer play. Because students often develop their conceptions of number with discrete, countable objects, developing thinking that supports this is important. In contrast to movements on a number line, thinking with a counterbalance conceptual model provides the opportunity to think about integers as "tangible." Within a counterbalance conceptual model, integers are conceptualized as two distinct quantities that neutralize. Ideas of neutralization are also important in mathematics from contexts like electron charges to areas beneath curves in calculus. However, as the students began this group session in the teaching experiment, the students did not appear to see the "neutralization" in the quantities. To emphasize this "neutralization" with a context, I decided to use a card game that uses integer cards from -8 to 8 (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014). I selected this card game because the cards are integer quantities that would remain present in their hands of cards, giving the students opportunities to experience neutralization and, consequently, the potential to develop the counterbalance conceptual model. For example, if a student had a hand of 2, -2, and 7, it was worth the same in this game as a hand of 3, -3, and 7.

Central to the notion of counterbalance, another goal included that children begin to make distinctions between the subtraction symbol and negative symbol (Gallardo & Rojano, 1994). For those reasons, after game play, the children were asked to make sense of fictitious children's hands of cards and write number sentences modeling the drawing or discarding of cards. This was done to help promote thinking and learning about both integer subtraction and the differentiation between the negative symbol (e.g., used when writing an integer in a number sentence) and the subtraction symbol (e.g., used when writing a number sentence for discarding). Some of the ways that these students engaged in these types of mathematical ideas during the integer play will be discussed in the following section.

Integer play with this card game satisfied Burghardt's (2011) essential criteria for play. This game provided the *opportunity for pleasurable experiences* because the children demonstrated excitement about playing the game and generally enjoyed playing with cards. This integer card game included *not fully functional* behaviors as the game was not necessary for survival but had the potential to satisfy the mathematical goals highlighted above. The game served as an activity *different from similar serious behavior*—the game included negative integers, which the children did not use during regular school instruction, and the children played the game outside of math class during their free time. The children played several rounds of the game and asked to keep playing after the game concluded—this illustrates *repetition* in play. *Initiated in the absence of stress*, the children volunteered to participate in this game play, which took place separate from formal instruction in a room outside of their classroom. The following excerpts will illustrate some of the mathematical goals achieved through this integer play and highlight the additional criteria of play achieved (Parks, 2015).

Integer Play: Addition of Integers

From the inauguration of the teaching experiment, the children illustrated an ability to add integers with success. Consequently, as the children engaged in the game play, they naturally added their cards with ease and did not initiate discussion about addition. In the first move of the game, Kim drew two cards:

Kim: Negative seven and eight.
Me: She had negative seven and eight. What do you think her point total is?
Alice: One.
Jace: One.
Me: Why do you all think it's one?
Jace: Because eight minus seven equals one.

All three students performed calculations repeatedly for the function of determining their scores. However, the students did not reference addition. Even when Jace needed to add two cards in this excerpt, -7 + 8, he interpreted this as 8 – 7, suggesting an interpretation of the negative as a subtraction sign (Bofferding, 2010). Rather than discussion about addition per se, the children's discussion focused on the discard of cards or how to get the largest point total, which often included making decisions between drawing a card (adding) or discarding a card (subtracting). This consequently resulted in children talking about situations where they were confronted with initial ideas of subtraction; they considered situations where it was better to discard larger negative integers from their card hands (subtraction) rather than to draw smaller negative integers to their card hands (addition), which was a mathematical expectation of this game (see, e.g., Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014). It was expected that the main opportunities for thinking and learning would be centered on the subtraction of integers and developing a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014), which are described next.

Integer Play: Subtraction of Integers

All three of the children discarded negative cards (e.g., -2) throughout the game and recognized that this increased the total points of their hands. The children successfully played ten rounds of the game, where each child confronted the option of discarding a card with a negative integer. Each child did this action and increased their point total; yet, the children did not necessarily explicitly recognize this physical action in the game play as subtraction. For this reason, at the end of the game, the children were asked to write number sentences representing some of their hands and fictitious children's hands of cards in order to see if they conceptualized discarding cards as subtraction. However, the children had difficulty writing number sentences. When the children began writing number sentences, they did not use subtraction for discarding cards; they would, instead, write the addition of the positive score. I prompted the children to think about how they could write number sentences that preserved the negativity of their cards. In the last minutes of the group's session, they began writing number sentences that involved the subtraction of integers. Jace and Alice worked together on writing a number sentence on a whiteboard together, while Kim observed, for discarding a -5 card from a fictitious student's hand of cards. The fictitious student's hand of cards included -3, -5, and 8, with a -2 card as an option to draw; therefore, they had decided it was best to take the current hand of cards (worth 0 points) and discard the -5.

Alice	(Writes 0 + 5 while whispering)
Jace:	(Whispers)
Alice:	Can you speak a little louder?
Jace:	Sure. I did zero (points at the 0 in 0 – -5 = 5) because that's what he had after the first problem. And then I did minus negative points (points at the "–" and then "-5") because he discarded the negative five and now he has five because there's not longer a negative five in the problem. In the first problem that he did. So that just adds five to it. Technically (gestures with fingers and makes "air quotes").
Alice:	(Looks at me) Well, I don't get how he got his answer of five.
Kim:	I don't get it.
Jace:	Alice, you're just doing what I did here (points at Alice's writing: 0 + 5).
Kim:	(Gets up out of seat and walks to the board where Jace and Alice are.)
Alice:	Yeah, but I don't get how he get got five.
Kim:	This was his first problem (circles Alice's number sentence before she simplified to find the initial point total: 8 + -3 + -5). And then this is the second problem (circles Jace's number sentence 0 – -5 = 5).
Alice:	Yeah, but I don't get how he got this answer (points at 5).

Kim wrote a number sentence with addition for discarding a -5 card. Although Jace was able to write a number sentence with subtraction and potentially make this connection at the end of this session, as Kim and Alice questioned him, he stated that he was confused too. In this excerpt, the children have generalized that their point total will go up by the absolute value of the negative they discard, a sophisticated observation. What remained was a matter of facilitating the children in connecting this generalization to subtracting a negative, which is an idea that may be developed later.

As the students shared this type of thinking about integers, they engaged in an *opportunity for social engagement* (Parks, 2015). Alice and Kim communicated to Jace their confusion, and Jace explained his thinking while they listened. This excerpt also illustrates *physical movement* and *use of materials that are appealing* (Parks, 2015). At first, Alice and Jace moved from the table to the whiteboard to discuss writing a number sentence, and then Kim followed. During the teaching experiment, the children often left the site where the cameras were to go write on the whiteboard. The whiteboard was an appealing department from their position at the table with paper and pencil. In general, when the students engaged in deep thinking together, they would move to this space, much like mathematicians around a chalkboard.

Given the challenges of writing a number sentence for the moves made in integer game play, subsequent sessions were developed to address subtracting negative integers. The challenges associated with the subtraction of integers lingered throughout the weeks of the teaching experience. The children's difficulty writing subtraction number sentences, but ease with discarding a negative and adding the amount to the deck, supports discussion about the difficulty of subtracting integers (e.g., Bofferding & Wessman-Enzinger, 2017) and supports research that demonstrates children's thinking is often different than adult's thinking (e.g., Bishop et al., 2014; Bofferding, 2014). However, this excerpt was included in demonstration of the initial thinking about integer subtraction that can happen during game play. Examining children's discussions during play experiences provides insight into their thinking, which may be supported later.

Integer Play: Counterbalance

From the beginning of the teaching experiment and throughout this group session, the children did not have difficulty with adding integers. However, despite their abilities to successfully add integers, the children did not all appear to draw on a counterbalance conceptual model. The counterbalance conceptual model involves children conceptualizing the addition of integers as integers that neutralize or balance each other out (Wessman-Enzinger & Mooney, 2014). In the following excerpt, Kim is faced with a decision to either discard a -7 card or to draw a +7 card. These moves have the same effect on her total points in her hand, and the children confront and reflect on this in the following excerpt.

Kim: It's the same, I think.
Alice: You could have … never mind.
Jace: No, because it would be zero, too.
Alice: I know something she [Kim] could do and it would make her score even higher, but… I'm not going to say it.
Kim: I don't think it could have.
Me: What do you think would make her score even higher?
Alice: If she picked this up (points at the 7 card).
Kim: I don't care really.
Jace: No, because she would still have the same amount.
Alice: Because she would have, then she would have, oh yeah… she would still have seven.
Jace: Yeah, because negative seven plus seven equal zero. So, should have still have…
Kim: Boom. Now I have eight points. Yay.

In this excerpt, Kim initially thought that discarding a -7 (subtracting -7) might be the same as drawing a +7 (adding 7). Alice thought that drawing a +7 card would make the score higher, than discarding a -7. The children discussed this. As part of that discussion, Jace provided the justification that 7 + -7 = 0—utilizing additive inverses is an important component when beginning to make sense of the counterbalance conceptual model. Jace reflected on this more than once, later during game play, stating:

> Alright, so. I have eight even though I have two eights in here. Actually, I have three if I count the negative eight. So… (writes on paper). Yeah, so I had an eight. I got a negative eight, so it's zero. So just got another eight and now it's eight.

Jace verbally recognized that $a + -a = 0$ in two instances during this group session. Although Kim and Alice did not verbally make those observations, they participated in the discussions where Jace shared this with them. Developing ideas about the additive inverses of integers is an important component to developing the use of the counterbalance conceptual model (Wessman-Enzinger, 2015; Wessman-Enzinger & Mooney, 2014). This excerpt highlights *creative thinking* (Parks, 2015) from Jace. Jace, without prompting from his peers or me, shared what he noticed about inverses. In this sense, Jace created this mathematics and shared his thinking about this observation. Although his peers did not ask him questions about his observations about inverses, his openness exposed Alice and Kim to this idea.

Integer Play: Minus Sign Versus Negative Sign

As the children engaged with integers through the game play, Jace highlighted that the role of a minus sign and negative sign is distinct (e.g., Bofferding, 2014)—a learning goal of the game with inclusion of the negatives on cards (use of negative sign) and discarding cards (use of minus sign when writing a number sentence). As the children wrote number sentences for representative hands of cards, Jace stated, "When you have a subtraction symbol (points at the 'minus' symbol) and a negative symbol (points at the negative number) you are just adding," referring to the number sentence 0 – -5 = 5. Kim, not convinced, stated, "Well, you are actually at zero." Jace responded, "If you take away a negative number that means that the negative number is no longer there. So like (starts writing on the board) five minus negative three would equal eight."

In this excerpt, Jace was trying to develop a rule for subtracting negative integers. For example, when Jace solved -7 + 8 in the previous section, he utilized 8–7 without discussion about this procedure. In this excerpt, Jace focused on the nuances of the sign and explicitly verbalized his procedure, but Kim and Alice were not convinced. Although it is noteworthy that Jace was trying to develop a rule or procedure for himself, through this discourse, he distinguished the negative symbol from the subtraction symbol. In this excerpt, the students focused on the minus symbol and the negative symbol. As they focused on the signs, treating the negative integer with its sign different than a minus sign represents the *use of materials that are appealing* (Parks, 2015). The students, prior to and during the teaching experiment, did not experience negative integers during their typical school day. In fact, according to *Common Core State Standards for Mathematics* recommendations (NGA & CCSSO, 2010), these students would not encounter subtraction of integers until 2 years later, and, as participants in this study, the students were attuned to this because they mentioned how they did not work with negative numbers during their typical school day and only within this teaching experiment. And, again, this negotiation on the differentiation of the role of the subtraction symbol and the negative symbol illustrates *opportunity for social engagement between the students* (Parks, 2015). This social engagement element of play was pivotal for addressing the mathematical goals of the integer play. In the past two excerpts, Jace verbalized a good understanding of concepts such as integer subtraction and symbol use. Through integer play, the students all engaged with these mathematical goals as they asked questions, discussed, and listened based on their understandings.

Although making sense of integer addition and subtraction, the counterbalance conceptual model, and differentiating the negative sign from the minus sign constituted the intended mathematical goals of the integer play, it was not important that the children mastered these ideas. Through conversation with each other, they were exposed to other ideas, like the ones that Jace presented in the past two excerpts that they had not played around with yet. Play is an ongoing activity that children use to help make sense of situations, and we cannot expect mastery immediately—especially with difficult ideas of integer subtraction. Providing opportunities for engagement with integer play is the point, because through play the children have the opportunity to work through different ideas and try new concepts out. Furthermore, the students thought about and engaged in other mathematics as they were playful with the integers in ways that were not planned by myself and the witness to the teaching experiment. The subsequent section highlights the robust mathematical ideas that may immerge when children play with integers.

Playing with Integers

The students engaged in integer play as they interacted in the game, *Integers: Draw or Discard*. Although immersed in integer play, the students played with integers in ways that occurred outside of the mathematical objectives of the game—*playing with the integers*. As the students created, wondered, imagined, and questioned with integers, they played with the integers. Three cases illustrating how the children played with the integers in this group session will be presented next. Two of cases illustrate the robust thinking and wondering they engaged in directly tied to integer addition and subtraction. Although the third illustration of playing with the integers does not connect to integer addition and subtraction, it connects to other advanced mathematical ideas. Each of these playing with integer cases will be linked to the work of mathematicians.

Playing with Integers: Order Versus Magnitude

Before the students began engaging in integer play, I explained the directions of the card game. I then asked the children who should go first. The following transcript illustrates the children playing with integers in this setting.

Me: So I was thinking... How do we decide who goes first though?
Kim: Rock, paper, scissors.
Alice: Or, who draws the highest card?
Kim: Yeah, draw highest card.
Jace: Yeah.
Me: Ok, so everyone takes...
Jace: Everyone takes one card and whoever has the highest.

(Alice, Jace, and Kim draw cards. Alice draws a -4 card, Jace draws a -8 card, and Kim draws a -7 card.)

Kim: I totally lost.
Alice: I did too.
Jace: I got negative eight.
Alice: I got negative four.
Me: Ok. And, you got what?
Kim: Negative seven...
Jace: So she goes first (points at Alice with -4).
Kim: (points at Jace with -8) So Jace's is the highest actually.
Alice: No, I am.
Jace: No, well...
Me: So, who is the highest?
Alice: (raises card in the air) Me!
Kim: Jace because his is the biggest in the negatives. Because we all have negatives, so.
Alice: Well, mine would be the biggest.
Jace: Well, she's the closest to one (pointing to Alice).
Me: So somebody said that they think Jace's is the biggest because it's negative eight.
Alice: (Shakes head no.)
Me: And, then Alice says no. So why did you think that Jace's is the biggest?
Kim: I don't know. They're all negative numbers and just like find out which one is bigger.
Jace: (Gasps.) I was wondering why you would want to discard cards. I'm like if they are all whatever why would you want to put one down. Ok, now I see.
Kim: Now I know why (holds the -8 card up in the air).
Me: And what's yours?
Alice: Negative four (holds up card).
Me: So which one do you think is bigger?

Alice: Mine.
Me: Why do you think yours is bigger?
Alice: It's closest to one. It's highest out of all of them.
Kim: Well, yeah.
Jace: Mmm-hmm.
Kim: So I'm second. I'm second (waves hands and card in the air) .

After this, I suggested that the children draw two new cards and start the game play. Although they never explicitly verbalized who should go first, Alice played first.

This excerpt highlights the elements of play: function, creativity, social engagement, and absence of stress. The children's suggestion of how to decide who should go first illustrated a *functional* component of playfulness (Burghardt, 2011); the students wanted to play the game and needed to decide who should go first, resulting in this mathematical discussion. This excerpt is playful because the children illustrated *creative thinking* (Parks, 2015); they created the ideas of order and magnitude when comparing integers. This excerpt is also playful because the children participated in *social engagement* (Parks, 2015); although the students did not verbalize a conclusive agreement on which card was "biggest," they decided to let Alice go first and played without conflict. The children freely had this discussion in the *absence of stress* (Burghardt, 2011); the children decided how they would determine who would go first in excitement to begin game play. During this freely chosen activity, the cards unexpectedly, and serendipitously, revealed all negative integers.

Distinguishing between order and magnitude of the integers is an important component of what it means to understand the integers and represents perquisite knowledge for integer addition and subtraction (Bofferding, 2014). Through deciding who should play the game first, the children played with the integers as they initiated a discussion about order and magnitude. Alice drew a -4 card; Jace drew a -8 card; and Kim drew a -7 card. The children found themselves in a situation grappling with order versus magnitude during the comparison of three negative integers: -4, -7, -8. Kim stated that -8 was "bigger" than the other numbers because -8 is "more negative"—employing magnitude-based reasoning (Bofferding, 2014). Alice and Jace reasoned that -4 is "highest" and "biggest" because it is closer to 1—employing order-based reasoning (Bofferding, 2014). Language issues of "bigger" and "higher" are also important tenants of the prerequisite knowledge that children need to make sense of as they begin to learning addition and subtraction (Bofferding & Hoffman, 2015).

As a society, we culturally emphasize order over magnitude with integer comparisons. That is, when comparing numbers like -4, -7, and -8, -4 > -8 is expected because of order, -4 is close to zero on the number line or -4 is more to the right on the number line than -8. However, often the work of mathematicians is magnitude based. That is, there are times when -8 is "bigger" than -4. For example, consider two velocity vectors, one with magnitude -8 and another with magnitude -4. The vector with a magnitude -8 would be considered "bigger." Also, this excerpt illustrates the children engaging in play that became an unresolved mathematical problem for them around order and magnitude. Sometimes mathematicians work on problems that are not resolved right away. This is the expected and normative work of mathematicians.

Playing with Integers: Permutations

Throughout the entire session, as the children played the integer game, they determined their total points in the game with the sum of the cards in their hand. Each of the children successfully wrote his or her total points on the recording sheet. However, throughout the entire session, the children would make jokes about having a point total that was different from what they were recording. The children physically moved their cards around on the table in different positions, using only cards with positive integers represented on them, to make "pretend" point totals. The excerpt of transcript below is from the first instance of this type of play in the session.

Alice: I have forty points. (Arranges cards 4 and 0 next to each other to look like 40.)

(Kim continues with game and draws a card.)

Kim: I will just take this one. (Takes cards and writes on recording sheet.)
Alice: Kim has like one hundred.
Kim: Nine.
Alice. Or, eighteen points. (Reaches over and touches Kim's cards, moving the 1 and 8 card next to each other.)

The children continued engaging in the integer play with the stated rules of the game; however, several times during this integer play, the children continued to arrange their positive cards, and notably not their negative cards, into different, "pretend" point totals. Although initiated by Alice, Kim did this later in the integer game play. Kim stated, "I made up thirty-eight and you guys are up in the eight hundreds"—referring to ordering the positive integer cards and notably not writing these point totals down. Alice and Jace participated in making permutations of their cards repeatedly as well. Looking at her hand that consisted of both positive and negative integers, she pulled the cards 0, 4, and 8 out of the hand. Discussing her actual point total, Alice whispered to Jace, "I have twelve. You have two more than me" and continued playfully, "I have eight hundred and four." Jace replied, "I'm going to lose. She has eight hundred and forty"—helping Alice make a larger valued number out of her current permutation.

This excerpt highlights elements of play: spontaneity, different from similar serious behaviors, repeated, creativity, and imagination. Without prompting the children engaged in extra, unplanned mathematics. The children played with the integers by making permutations with their positive integers *spontaneously*—an element of playfulness (Burghardt, 2011). This excerpt is also *different from similar serious behaviors* (Burghardt, 2011); in fact, the children attuned to this difference and did not record these "pretend," permutated scores on their recording sheet. This excerpt is playful because it illustrates the children engaging in an act that was *pleasurable and lighthearted* to them (Burghardt, 2011); the children treated these permutations as pretend scores as they continued with the expected directions of the game and recorded different point scores than they verbally stated with the permutations. The children *repeated* this type of play throughout the session (Burghardt, 2011). This example is also playful because it illustrates the *creative thinking* and *imaginations* of the children (Parks, 2015); they created this play with permutations and imagined larger scores than they actually had based on the rules of the integer game.

The children constructed permutations with the positive integers only. They ordered their positive integer cards, utilized the place value system, and made new point totals from the permutation that would give the largest positive number. The children implicitly recognized that the base-10 system utilizes positive digits in the place value system, rather than negative digits. That is, if you have -1 and -8 cards, they were more than likely not permutated because -1 and -8 are not utilized as digits to make numbers. In order to use the negative cards, the students would have needed to take a negative card and place it first, like -1 and 8 to make -18 or 8 – 1. However, they did not do this. In addition to constructing permutations with the positive integer cards, they reasoned about what permutation provided the largest positive number. In this sense, as the children played with the integers, they also played with the idea of permutations. Although permutations are an important mathematical concept, it is not explicitly needed prerequisite knowledge for the teaching and learning of integer addition and subtraction. This is a consequence of the freedom of play; without prompting, the children engaged in extra mathematics. Although not a mathematical goal of original integer play, the children fearlessly played with integers in a mathematically productive way.

The ways that the children played with the integers in this excerpt mirrors the ways that mathematicians play with numbers as well. Similar to the work of the children in this excerpt, mathematicians engage in recreational mathematics (see, e.g., *Journal of Recreational Mathematics*). Some mathematics is simply for the joy and interest of doing mathematics (e.g., logic puzzles, happy numbers, star tangrams). In fact, often within the domain of recreational mathematics, permutations or combinations with integers are necessary. For example, pentominoes are common puzzles accessible to children but are also the basis for some interesting recreational mathematics (see, e.g., Golomb, 1994; Wessman-Enzinger, 2013). A pentomino is created by permutations of the five unit squares in such a way that each square touches another square on at least one side—creating 12 pentominoes. Some recreational mathematics topics have included creating twin pentomino towers (e.g., stacking pentominoes vertically, creating the same-shaped towers with different pieces). Although the children's play did not directly relate to integer addition and subtraction, the children did play with integers through permutations—a mathematically substantial way linked to the work of mathematicians (see, e.g., Knuth, 2000).

Playing with Integers: Zero

After the children played ten rounds of the game, they were shown various hands of cards from fictitious children. Alice, Jace, and Kim considered these hands of cards, played with their physical cards, and decided what move the fictitious children should make. The children also wrote number sentences for the point totals of the various hands when drawing or discarding cards. As the children attempted to write a number sentence for a hand of cards, Jace posed a question.

Jace: I have a question. Would zero count as a negative number?
Me: Do you think that zero would count as a negative number?
Alice: No.
Kim: Hmm... No.
Jace: Well, it's not a whole number.
Kim: I think it would actually equal both.
Me: You think it would equal both?
Kim: I mean it would be both. (Shakes hand side to side).
Alice: I think it's kind of in the middle.
Jace: Because zero is nothing.
Me: Hmmm.
Jace: And, negative numbers are nothing. But, it doesn't have a negative symbol in front of it.
Alice: Zero's like not a number because it's nothing.
Jace: Well, so is negative numbers.
Kim: (Laughs.)
Alice: Yeah, but they're something.
Jace: My mind is blown.
Alice: (Laughs.)
Kim: Zero is sort of important. It's like the line below the whole numbers to let you know when you are starting the negatives.
Alice: I think the answer for this one (points at the sheet of paper, returning to the trying to write a number sentence for a hand of cards) is five, but I don't get my number sentence.

The children grappled the nature of zero in this excerpt. They initiated a discussion about whether zero is negative or not. In addition to discussing whether zero is negative or not, Alice wondered if zero is not even a number, which then prompted Jace to reflect on the physical embodiment of the integers, stating that "negative numbers are nothing" also. Children often have misconceptions about zero (e.g., Bofferding & Alexander, 2011; Gallardo & Hernández, 2006; Seidelmann, 2004), and making sense of zero as neither a positive nor negative number is important. Recognizing that zero is neither positive nor negative is a component of highlighting the symmetry of the negatives with zero as the center.

This excerpt highlights elements of play: spontaneity, imagination, social engagement, creativity, and stress-free initiation. This excerpt is playful because Jace *spontaneously* asked a question about whether zero is negative, also highlighting his *imaginative thinking* about the integers (Burghardt, 2011). Also illustrating playfulness, the children engaged in *social engagement*, considered Jace's question, and shared their opinions (Parks, 2015). This excerpt is also playful because the children illustrated *creative thinking* (Parks, 2015); they thought that maybe zero was not a number, maybe zero was both positive and negative, or maybe zero was just a number in the middle. Illustrating an *initiation in a stress-free environment*, in a freely chosen discussion, Alice decided to transition from this conversation back to the task of writing a number sentence (Burghardt, 2011).

The ways that the children contemplated the nature of zero in this excerpt mimics the historical struggles mathematicians faced as they made sense of zero as well. Gallardo and Hernández (2006) wrote about this, "Piaget (1960) states that one of the great discoveries in the history of mathematics was the fact that the zero and negatives were converted into numbers" (p. 153). Historically, mathematicians have also grappled with similar ideas about the nature of zero (Kaplan, 1999), and these children did as well through their wonderings of the positivity and negativity of zero.

Discussion

This chapter described both instances of integer play and playing with integers within a specific group session of a teaching experiment on integer addition and subtraction. Describing instances of integer play (e.g., a game with integers) and playing with integers (e.g., contemplating the negativity of zero) that children and students engage in is important in order to facilitate these types of play in the future. Although the descriptions of integer play and playing with integers in this chapter come from a specific instructional experience designed for integer addition and subtraction for Grade 5 students, these instances specify the rich creativity and meaningful mathematics that children play with. Not only do these instances of play highlight robust mathematics of children connected to the work of research mathematics, but integer play is a way to share integer instruction earlier than recommendations, and playing with integers is a way to prolong play in school and can also serve as a way to provide equitable instruction for children.

Integer Play as a Way to Bring Integers to Curriculum Sooner

We are situated in an era where research illustrates that young children are capable of reasoning about integers (e.g., Bofferding, 2014); yet, standards do not suggest instruction with integers until later grades (NGA & CSSSO, 2010), and most curriculum in the USA supports this as well (Whitacre et al., 2011). Illustrating instances of integer play and playing with integers may provide an outlet for bringing thinking and learning with integers to earlier grades. Although it is not novel to suggest integer instruction earlier (see, e.g., Bofferding, 2014), current recommendations currently maintain integer operations in Grade 7. Yet, Bofferding and Hoffman (2015) illustrated that children are capable of engaging with integers, as young as kindergarten, in game play, and this type of game play is productive in developing conceptions of numbers.

Why Integers? Although Grade 5 is not much sooner than recommendations in standards (e.g., NGA & CCSSO, 2010), even playing with integer operations 2 years prior to formal instruction will be beneficial to break generalizations formed by whole numbers (e.g., adding always makes larger, Bofferding & Wessman-Enzinger, 2017). Children are capable of many things, but there should be a focus on integers in elementary school to confront misconceptions of working with only positive integers. As illustrated in both this chapter and entire book, by working with integers, children confront the ideas that:

- Addition does not always make the sum "larger."
- Subtraction does not always make the difference "smaller."
- "Larger" and "smaller" have different meanings with order-based and magnitude-based reasoning when extending beyond positive numbers.
- The number line does not just extend infinitely in only one direction.

Because the physical embodiment of the negative integers is not as natural as the counting numbers (e.g., 1, 2, 3, …) or positive real numbers (e.g., 1/2, 0.4), there is something inherently playful with the integers that is due to its challenging nature compared to other numbers. By engaging in work with integers, children potentially gain a deeper understanding of the number systems they are, by standard recommendations, supposed to learn. As illustrated in this chapter, the children also gain more than that when working with integers—they gain experiences of thinking like a mathematician as they create uses of integer operations, make sense of magnitude- and order-based reasoning, or even make permutations of positive integers.

Yes, we need to teach operations with whole numbers and positive integers and positive rational numbers as the standards recommend. But, is that truly possible when we are potentially generating and establishing deep misconceptions (e.g., subtraction always makes smaller)? Not only do we need to utilize integer play and utilize it sooner than recommendations, but we also need to allow for children to play with integers and examine the ways that children play with integers as they engage in this type of play.

Integer Play and Playing with Integers as a Way to Prolong Play in Schools

Parks (2015) shared the importance of incorporating play beyond early childhood—suggesting that even children in Grades 2 and 3 should have time set aside for play. Featherstone (2000) illustrated in a Grade 3 classroom that the use of negative integers opened a space for imaginative mathematical play in the classroom. The instances presented in this chapter of children *playing with integers* illustrated more elements of play than even in the *integer play* section. As the children played with integers, they enjoyed their creative mathematics, which included extra mathematics than the planned mathematical goals of the game. For example, as the children made their permutations of positive integers, they were joking with each other. They laughed, spoke in silly voices, and did not take their permutated score seriously. As they discarded negative integers and made sense of zero scores, they laughed and teased each other around a ficitious game with pretend scores.

Alice: I have eight thousand and ... (Alice making a joke as she permutated her positive cards.)
Kim: (Takes the card from the center pile and writes on the paper). Oh my god, I will just have to add it. Now I have negative fifteen. Sad day.
Jace: (J flips the center card.) Oh my god!

(laughter)

Jace: (Discards his final card.) I hope you guys are happy. I have nothing. Wait no, I should take that one. Now, Kim you are in second place.
Kim: (Claps hands together) Woot!

When the game ended, the children expressed continued joy about engaging in this play by asking to continue to play.

Jace: We are the champions.
Kim: Do you have another one (holding up a recording sheet)?

This points to a twofold implication centered on prolonged play in school. First, utilizing games in later elementary grades, when typical conventions of play may not have as a prominent of a role, is one way to prolong play in schools. While the use of game play does not necessarily dictate play (e.g., a game on multiplication facts will likely not have the same results), integer play and playing with integers offer enough imagination and challenge to support authentic play. Second, playing with integers effectively engages students in mathematics at a time when many children seem scared of it—providing a space for children to be fearless and creative in mathematics.

Playing with Integers as an Equity Tool

With integer play, teachers determine the play and set the mathematical goals. I, for instance, planned to use an integer game (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014) and started the group session with predetermined mathematical goals. In the selected excerpts highlighted in this chapter, Jace appeared to conceptualize the integers in these intended ways and explained this reasoning to his peers, Alice and Kim. However, when playing with integers, the students set the agenda and determined what mathematics would be explored. In the integer play, Jace seemed to shine: noticing inverses and differentiating the use of the minus symbol from negative symbol. But, when playing with the integers, other students brought their mathematics to the table. Alice and Kim questioned the role of zero and compared the nuances in order and magnitude, a goal I did not plan.

Playing with integers not only provided opportunity for earlier integer instruction and prolonged play in school but also provided an equitable opportunity for all students to be successful mathematically. Because of the freedom of playing with integers, rather than just integer play, the children entered the play and mathematics in their own way, freely sharing their creative, playful, and valuable ideas—like permutations. Providing space for playing with integers is a pedagogical tool for equitable practices in school mathematics.

Integer Play and Playing with Integers as a Space for Future Research

The children created, invented, and played with the integers—this is the beauty of games. With integer play and playing with integers, there are opportunities for unlimited mathematical experiences—the children in these excerpts played with more mathematics than planned in the intended mathematical goals of the game. As researchers and educators, we want to pick games where this potential for playing with integers is large, and the only way we can know that for sure is by studying them. Then, if additional opportunities for mathematics arise, we can modify the games to encourage it more. For example, a revised version of the game could require that whoever draws the largest card has to go first to encourage more debates about order and magnitude like Alice, Jace, and Kim engaged in.

Conclusion

These instances of integer play highlight that children are capable of thinking about integer addition and subtraction. Through integer play, children encountered opportunities for playing with integers in novel ways. The excerpts of playing with integers illustrate the playful curiosities arising out of integer addition and subtraction that tended to be concepts that we think of as "prerequisite knowledge" (e.g., magnitude or order, sign of zero). Yet, students also began developing integer knowledge that is more nuanced for integer addition and subtraction (e.g., how negatives and positives can "balance" each other) during integer play. Because the children demonstrated capability in solving some integer addition and subtraction problems in this session and throughout the teaching experiment, these examples of integer play and playing with integers highlights that learning about typical prerequisite knowledge (e.g., order, magnitude, use of minus sign) may be developed in tandem with integer addition and subtraction. Furthermore, not only did the children engage in thinking about addition and subtraction of integers, as well as other integer concepts, the children engaged in the work of mathematicians. As children played with the integers and engaged in the work of young mathematicians, they did the thinking and learning most important to integers: imaginative and creative play.

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