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Integers as Directed Quantities (Chapter 13 in Constructing Number)

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Integers as Directed Quantities

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Abstract Mathematics education researchers have long pursued—and many still pursue—an ideal instructional model for operations on integers. In this chapter, I argue that such a pursuit may be futile. Additionally, I highlight that ideas of relativity have been overlooked; and, I contend that current uses of translation within current integer instructional models do not align with learners' inventions. Yet, conceptions of relativity and translation are essential for making sense of integers as directed quantities. I advocate for drawing on learners' unique conceptions and actions about directed number in developing instructional models. Providing evidence of student work from my research, I illustrate the powerful constructions of relativity and translation as students engage with directed quantities.

Keywords Conceptual models · Integers · Integer addition and subtraction · Integer instructional models · Integer operations · Number line

13.1 Introduction: Pursuit of the Ideal Instructional Model for Integers

The perfect model for teaching and learning operations on integers is the holy grail of integer research in mathematics education. After taking over 1500 years to formally account for integers (e.g., Henley, 1999), mathematicians and educators have sought the perfect model for integer operations through various contexts, including the number line (e.g., Heeffer, 2011; Schubring, 2005; Wessman-Enzinger, 2018a). Yet, the use of the much-vaunted number line broke down for nineteenth century mathematicians for the operations of multiplication and division (Heeffer, 2011). Centuries later, even our social media is proliferated with math teacher chats, groups, and tweets posting and discussing their wonderings about instructional models for integers. For example, a recent post in a large Facebook group of

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mathematics education teachers and researcher quickly welcomed 33 different comments (not including replies) on the following question (Mathematics Education Researchers, 2017):

> Does anybody know a good model of negative integer (number) operations? I am working with middle school teacher and students. It's hard to find a visual model that illustrates the meaning of negative number division/multiplication such as 10/−2. Is there a good explanation that makes sense of the operation with a good connection with the division topic?

This holy grail—the ideal model for teaching and learning of integer operations—will always remain mythical. We will never find the perfect instructional model for all integer operations, despite our commitment, because all models for integers break down at some point (Galbraith, 1974). Although all models have affordances and limitations (Vig, Murray, & Star, 2014), integer instructional models have particular limitations (Peled & Carraher, 2008) because of the constraints on the physical embodiment of negative numbers (Martínez, 2006).

For a few instructional models, some of these affordances and limitations are highlighted in Table 13.1. In an article about mythical creatures in *The New Yorker*, Schultz (2017) commented, "One of the strangest things about the human mind is that it can reason about unreasonable things" (para. 2). Despite the affordances and limitations of instructional models for integers, we can still reason about them. Perhaps this is why an ideal instructional model for integers has been pursued so vigorously.

Table 13.1 highlights only a diminutive portion of the different instructional models that have been proposed across decades of integer research (e.g., Bruno & Martinon, 1996; Janvier, 1985; Liebeck, 1990; Linchevski & Williams, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988). There will always be a hunt for the ideal integers instructional model. This pursuit of the unattainable is not so uncommon: "The relative plausibility of impossible beings tells you a lot how the mind works" (Schultz, 2017). It seems that an ideal instructional model for integer operations might exist, for we know that robust models exist for whole number operations. Furthermore, studies with instructional models for integers make the existence of such a model seem plausible because these investigations provide interesting results and insights into students' thinking (e.g., Bofferding, 2014; Tsang, Blair, Bofferding, & Schwartz, 2015). Consequently, math educators and psychologists will continue to pursue better instructional models for integer operations (e.g., Moreno & Mayer, 1999; Pettis & Glancy, 2015; Stephan and Akyuz, 2012; Tsang et al., 2015).

Yet, we can do better than pursuing an ideal integer instructional model. In this chapter, rather than presenting more top-down integer instructional models, I instead point to how conceptualizing integers as directed quantities is a powerful conceptual tool. We should focus on the constructions of learners and the integer models they create prior to their use of integer instructional models made by adults. It is notable that many of our instructional models (those formed by adults) incorporate ideas of movement and measurement metaphors (Chiu, 2001; Lakoff & Núñez, 2000), which align to larger mathematical ideas. This is likely because our students naturally employ ideas of movement and measurement; yet, we need to understand what learners' integer constructions around movement and measurement for inte-

Table 13.1 Affordances and limitations of some instructional models for integer operations

Instructional model(s)	Sample reference(s)	Affordances	Limitations
Two-colored chips	Liebeck (1990), Murray (2018), Vig et al. (2014)	The use of two-colored chips builds on children's experiences with discrete, physical objects. These types of models work well with integer addition (e.g., -2 + 3 = □). These types of models also work well for integer multiplication and division, where one factor is a negative integer and one factor is a positive integers (e.g., -2 × 3 = □, 6 ÷ -2 = □).	These types of models are not intuitive for integer subtraction because "zero pairs" must unnaturally be added into set of two-colored chips (e.g., -2 − 3 = □). These types of models do not work for the multiplication of negative integers or the division of two negative integers, unless extra chips are available to imagine taking away chips. The use of two-colored chips may be used differently and does not inherently dictate a particular instructional model.
Traditional debt and credit contexts	Wessman-Enzinger and Mooney (2014), Whitacre et al. (2015)	Debts and credits exist in the world, and students might connect the integers to related contexts. Learners may think about debts and credits in relation to integer addition and subtraction in ways that do not involve traditional notions of money (e.g., owing candy bars, lost pencils).	Although we apply negative integers to debts, we do not have to. Even secondary students do not necessarily connect traditional debt and credit contexts naturally to the negative integers.
"Net worth" context paired with empty number line	Stephan and Akyuz (2012)	The "net worth" is used in ways that emulate counterbalancing of quantities, which is natural for learners. "Net worth" modifies traditionally used ideas of debts and credits, which are prevalent in standards and curricula. "Net worth" is an intuitive space for learners to make sense of integer addition and subtraction.	The particular instructional model that Stephan and Akyuz (2012) use is a blended model, paring "net worth" with empty number lines. "Net worth" by itself does not naturally dictate use of an empty number line (as no linear movement is inherently a part of it). When "net worth" is paired with the use of the empty number line, learners use the change/displacements only on the number line, which is slightly different than "net worth" where all of the quantities remain present. This instructional model works well for addition, subtraction, and multiplication. It does not work as well for division (Stephan, personal communication, April 10, 2018).

(continued)

Table 13.1 (continued)

Instructional model(s)	Sample reference(s)	Affordances	Limitations
Movements on a number line	Nurnberger-Haag (2007)	This model incorporates use of the number line and critiques of other number lines that are built on sets of rules and procedures. This instructional model is designed for all four integer operations. This model support physical embodiments of integer operations and students will physically move as they make sense of integer addition and subtraction.	Ultimately, this instructional model (despite critiques of other models) also builds on its own set of rules and procedures that are not necessarily intuitive or invented by learners. "Adding and Subtracting Integers: (Remember to add or subtract only two numbers at a time.) 1. Start on a number line at the first number of the problem 2. Always start with a positive attitude! (Face the positive direction.) 3. Turn the ______ direction for every '–' sign after the first number. Whatever direction you end up facing is the direction you will walk 4. Walk the number of steps indicated by the absolute value of the second number." (p. 119) Like the two-colored chip model above, there are different interpretations of "walking" on a number line.
Folding number line	Tsang, Blair, Bofferding, and Schwartz (2015)	This model capitalizes on evidence in cognitive research that humans are drawn to symmetry and supports work in embodied cognition that we think about things we physically experience. This model supports the conceptual development of symmetry and works quite well with the addition of integers.	Extending this instructional model beyond integer addition is complicated, if not impossible.

gers look like. Conceptualizing integers as directed quantities, with movement and measurement, requires mathematical ideas of translation and relativity.

13.2 Definitions of Relativity and Translation

The conceptualization of integers as directed quantities requires using integers as a relative number (Gallardo, 2002; Thompson & Dreyfus, 1988). The starting point and directions that are attributed as positive and negative numbers are arbitrary, even if intentionally determined making integers inherently relative. *Relativity* entails using the integers as comparative numbers or relative numbers (Wessman-Enzinger, 2015). The integers describe relative positions. Zero represents the point of reference, which may be intentionally or arbitrarily selected. Distinctively, the zero does not represent a quantity of nothing, but is treated as a referent for comparison, as one reasons about integers with relativity.

The conceptualization of integers as directed quantities includes both movement and measurement as operations with integers are performed (e.g., Bofferding, 2014; Chiu, 2001; Lakoff & Núñez, 2000; Thompson & Dreyfus, 1988). These ideas of linear movement point to conceptualizations of translations. Translation entails using integers as vectors (Wessman-Enzinger, 2015). Integers are often treated as vectors moving right or left or up and down a linear model, coordinate plane, or three-dimensional space. Zero may be conceptualized as a vector or a translation of no movement. Similar to conceptualizations of relativity, the zero can also represent any arbitrary point with the addition and subtraction of positive and negative numbers representing the translation in one direction or another from the relative zero (Thompson & Dreyfus, 1988).

When conceptualizing integers with translation, distance may be used without direction specified, called *absolute value* (Wessman-Enzinger & Bofferding, 2018); for example, the distance between -2 and -3 is 1 (going from -2 to -3, or -3 to -2). Although it is possible to conceptualize distance without direction, it is still considered to be drawing upon translation because all distance must be conceptualized with direction at some point. When the direction of the distance is explicit, allowing for negative distances, this is called *directed value* (Wessman-Enzinger & Bofferding, 2018); e.g., the distance from -2 to -3 is -1 and from -3 to -2 is 1. Moving in "more" and "least" negative (or positive) directions support use of directed value (Bofferding, 2014; Bofferding & Farmer, 2018). Translation may also be employed with the use of counting strategies because counting fundamentally draws on movement and order (Bofferding & Wessman-Enzinger, 2018; Wessman-Enzinger, 2015).

These definitions of translation and relativity describe two broad types of conceptualizations that learners construct as they engage with integer operations. Learners' constructions of relativity and translation are powerful conceptual tools for making sense of integers as directed number. We should focus more on the conceptual tools learners construct within instruction rather than top-down integer instructional models.

13.3 Directed Numbers as a Powerful Conceptual Tool

> *You have the negatives like a thought thing. It's kind of mental. And, you can like literally take away so many apples or slices of pie from someone and you can still have it. And the other person would still end up having some. Whereas, negatives, if you have something and you take something away from them and they don't have any, you can still keep taking more. But, you don't really have anything. You still won't.* (Drake, Grade 8)

In this excerpt, we see Drake, a student with 3 years of experience operating with negative integers, struggling with the abstract nature of the physical embodiment of these numbers. Negative integers, as Drake points out, cannot be physically modeled with discrete objects in our world and are abstracted mathematical objects. Although all numbers are abstract, learning about the negative integers demands a different realm of abstraction (Fischbein, 1987).

Learners use manipulatives or hands-on activities as they learn whole numbers and fractions (e.g., Martin & Schwartz, 2005; Moyer, 2001; Siegler & Ramani, 2009). For these reasons, mathematics educators might think that one affordance of using physical objects with the teaching and learning of integers is that learners draw upon something familiar (e.g., Bolyard & Moyer-Packenham, 2006). Embodied cognitive scientists and psychologists also recognize that our experiences and actions impact our thoughts (e.g., Barsalou, 2008; Goldin-Meadow, Cook, & Mitchell, 2009; Lakoff & Núñez, 2000; Tsang et al., 2015). Yet, there are obstacles when extending previous experiences with whole number and physical objects to negative integers; negative integers are not naturally extended in the physical realm and have limitations in physical embodiment (e.g., Peled & Carraher, 2008; Martínez, 2006). Negative integers, for instance, have to be mapped to the physical objects representing them. For example, the use of two-colored chips, or a cancellation model, is one way that integers are represented with physical objects, where the negative integers are represented by red chips and positive integers by black chips (e.g., Liebeck, 1990). A negative integer, -*n*, is modeled with *n* objects that need to be physically present and countable. Then, -*n* is represented, by extension, with each countable object representing -1. A consequence of this type of modeling with physical objects is that some problems, such as 2 − -1, may not be intuitive and modeling them with physical objects can be challenging (Bofferding & Wessman-Enzinger, 2017; Vig et al., 2014).

Consequently, inaugural learning experiences with integers need to overcome traditional notions of the physical embodiment of number. Specifically, these learning experiences need to support the transition from discrete and static ways of thinking about number to thinking about number as continuous directed quantities. One way to so is to provide learners with opportunities to *create* their own models, rather

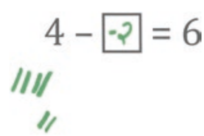


Fig. 13.1 Alice's drawing of discrete objects that supports transitioning from discrete to continuous objects

than giving them instructional models. Learners may create models that bridge discrete and continuous representations of integers (see, e.g., Fig. 13.1).

Figure 13.1 illustrates work from a Grade 5 student, Alice, who drew two sets of discrete objects, 4 tallies to the left, and 2 tallies to the right when solving 4 − □ = 6.

Alice: [Draws four tally marks. Thinks for a bit and draws two more tally marks lower and to the right. Then writes -2 in the box.] I did four minus negative two and I got six because … I did four right here (points to upper tallies) and two (points to lower tallies). And, then this is six.
Teacher-researcher: How did you know it was -2?
Alice: Well, because I did two… I did it backwards (moves pen across 4 − -2 = 6). If I did two plus four I got six. So, then I thought it would be negative two.
Teacher-researcher: What do you mean by backwards?
Alice: If like six (points at 6) minus two would give you four [6 − 2 = 4]. So, I thought four minus negative two would give you six [4 − -2 = 6].

Alice used additive inverses, changing 4 − □ = 6 to 6 + □ = 4. She used 6 − 2 (instead of stating 6 + -2) when she solved this. Building on her discrete representations, she made analogies to whole number addition and subtraction (e.g., "working backwards," comparing to 6 − 2). Her representation of discrete objects, paired with addition and subtraction, points to potential for developing notions of directed number. Instructional experiences could connect Alice's invented reasoning to her drawing. A teacher could ask, "In what ways is Alice's drawing related to her strategy?" Then, her drawing could leverage ideas of movement; that 4 − □ = 6 and 6 + □ = 4 can represent equivalent situations. Or, her representation could be built upon and turned into a continuous model (e.g., her tallies can be related to spaces on a number line).

As learners transition from thinking about whole number operations to integer operations, a wealth of significant conceptual changes need to occur (Bofferding, 2014). As Drake's excerpt above illustrates, learners need to transition from physically operating with number to "thought things." Some of the potential challenges of transitioning from thinking about whole numbers to integers are highlighted below:

- Whole numbers can be physically embodied naturally with counting objects (e.g., Smith, Sera, & Gattuso, 1988); integers have limitations with physical embodiment, especially with counting physical objects (Martínez, 2006; Lakoff & Núñez, 2000).
- Whole number units are positive (Steffe, 1983); integer units are positive units or negative units.
- Whole number direction is one-directional; integer direction is two-directional (Bofferding, 2014).
- Whole numbers have similar order and magnitude, $2 < 5$ and $|2| < |5|$; integers have different order and magnitude, $-2 > -5$ and $|-2| < |-5|$ (Bofferding, 2010, 2014; Wessman-Enzinger, 2018a, c).
- Integers are relative numbers (Gallardo, 2002) in ways that only positive numbers are not.

Engaging with directed number as an inventive, playful "thought thing," outside of pre-determined instructional models, may help learners make these transitions

(Bofferding, Aqazade, & Farmer, 2018; Wessman-Enzinger, 2018b, c). Directed quantities—an inherent part of making sense of integer operations (Poirier & Bednarz, 1991; Ulrich, 2013; Thompson & Dreyfus, 1988)—is a rich place to enter discussion about what thinking about integer operations entails: relativity and translation. Although thinking and learning about integers as directed quantities may have challenges, I argue that conceptualizing integers as directed quantity offers more than any singular instructional model. The following sections delineate some of the ways children construct directed quantities through the lens of the mathematical ideas of relativity and translation. Learners make sense of directed number in powerful ways (e.g., Bofferding, 2014; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016), negating the need to find the mythical, perfect instructional model for integer addition and subtraction. Examples of student work later in this chapter highlight that if we build on learners' created models, rather than giving top-down models, there is no longer a need for a singular, ideal integer instructional model. The tenants of directed number, like translation and relativity, are often overlooked in descriptions of children's thinking; examining these specific ways of thinking can provide insight into the most robust types of models. Yet, if we build on learners' constructions of number as the "instructional model" instead, then we need to describe their constructions of translation and relativity in more depth.

In the following sections, I describe translation and relativity as components of understanding ways learners construct directed quantities. Specifically, I address the following points:

1. There are rich historical backgrounds that support the conceptualizations of translation and relativity of integers; as a society we grappled with ideas of translation and relativity for centuries.
2. Existing research highlights the capabilities and thinking of learners as they engage with integer addition and subtraction; yet, how learners construct ideas of relativity is underrepresented.
3. Many different contextual situations and problem types support different ways of thinking about translation and relativity of integers; one instructional model alone cannot fulfill these needs.
4. Children create powerful ways of thinking with translation and relativity that are significantly different than traditional integer instructional models. Children's unique constructions will point us in better directions for thinking and learning in instructional spaces.

13.4 Coordinating Relativity and Integers as Directed Quantities

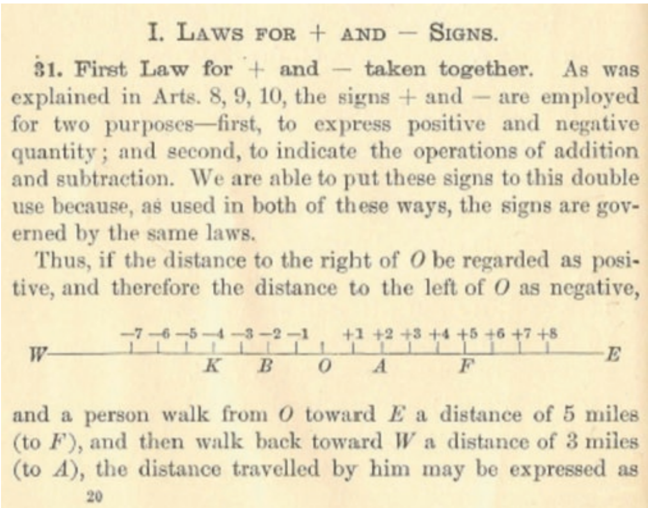
The idea of relativity is a mathematical concept that extends itself beyond integer operations (e.g., choosing to use a Cartesian coordinate plane or polar coordinate plane is an example of relativity). In this section, I discuss the idea of coordinating relativity with integer addition and subtraction as a conceptual tool to make sense of integers as directed number.

13.4.1 A Historical Perspective of Relativity

Nineteenth century North American arithmetic and algebra texts first present integers as relative numbers (Wessman-Enzinger, 2018a). Some of the first illustrations of the number line in North American school mathematics arithmetic and algebra texts included a relative number line. It is worth noting that the relative number line in Fig. 13.2 does not highlight zero, but rather a point “O,” where numbers to the right are positive numbers and numbers to the left are negative numbers.

The mathematical concept of relativity, which is foundational for using integers as directed number, evolved over time (Wessman-Enzinger, 2018a). Our modern definition of integers, with integers as a subset of the real numbers and rational numbers, prioritizes the integers as objects and overlooks relativity. Current curricula and standards do not support extensive time to build the integers conceptually (e.g., integers are often not in the elementary curricula). Our modern curricula and standards, in fact, omit ideas of relativity with integers. Yet, without relativity traditional instructional cancellation models do not work well with subtraction. Consider, for example, illustrating 2 − -3 with a two-colored chip model. One needs to use relativity to represent 2 in multiple ways in order to remove -3 chips.

Our modern curricula and standards even treat the integers as though they are fixed objects on a number line (e.g., negatives must always go on the left side of the number line). These types of ideas, such as negative integers being placed anywhere


I. Laws for + and − Signs.

31. First Law for + and − taken together. As was explained in Arts. 8, 9, 10, the signs + and − are employed for two purposes—first, to express positive and negative quantity; and second, to indicate the operations of addition and subtraction. We are able to put these signs to this double use because, as used in both of these ways, the signs are governed by the same laws.

Thus, if the distance to the right of *O* be regarded as positive, and therefore the distance to the left of *O* as negative,

and a person walk from *O* toward *E* a distance of 5 miles (to *F*), and then walk back toward *W* a distance of 3 miles (to *A*), the distance travelled by him may be expressed as

20

Fig. 13.2 Illustration of a relative number line in Durrell and Robbins (1897, p. 20) on the number line (e.g., negatives on the right side instead of the left side), are absent from curricula and standards. Our standard documents (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) do not highlight the relativity of integers. The implication is that learners must implicitly think about and use relativity; yet, relativity is essential for constructing directed number. Learners need time to conceptualize and build their meanings of relativity, which they might do naturally if allowed to construct models for themselves.

Despite the lack of emphasis on developing relativity in modern standards and curricula, researchers have reflected on it. Gallardo (2002) points to different understandings of integers, making explicit that one of those includes recognizing integers as relative numbers. Carraher, Schliemann, and Brizuela (2001) reflected on an N-number line, where the ordering is centered on N (e.g., $N - 3$, $N - 2$, $N - 1$, N, $N + 1$, $N + 2$, $N + 3$). A distinguishing element of this N-number line is that N is unknown and could be represented by any number, thus incorporating the idea of relativity. The N-number line presented by Carraher et al. captures the essence of "relative numbers" and "relative number lines" found in early arithmetic and algebra texts in the nineteenth century (see, e.g., Durell and Robbins, 1897; Loomis, 1857). For example, Loomis (1857) began his introduction of the negative integers by describing the order of the negative integers through the context of the thermometer. After discussing the thermometer and ordering, Loomis commented on relativity in reference to contexts beyond temperature:

> It has already been remarked, in Art. 5, that algebra differs from arithmetic in the use of negative quantities, and it is important that the beginner should obtain clear ideas of their nature. In many cases, the terms positive and negative are merely relative. They indicated some sort of *opposition* between two classes of quantities, such that if one class should be added, the other ought to be subtracted. Thus, if a ship sails alternately northward and southward, and the motion in one direction is called positive, and the motion in the opposite direction should be considered negative. (pp. 18–19)

In this description, the integers are described as a relative number, where two directions are provided in "opposition" from an arbitrary referent.

13.4.2 A Contextual Perspective of Relativity

> *Say you are down five runs in the first inning of a baseball game. And you end up losing by fifteen runs. You would have to have ten runs in the other innings to be down by fifteen runs.* (Joseph, Grade 8, -5 + □ = -15)

Joseph, posing a story for -5 + □ = -15, makes use of integers as relative numbers with an unknown referent. When Joseph posed this story for the first time, I remember initially thinking this was quite a novel context—and then, I reflected on the mathematics he employed. What is the score of the game? Although the score of the game is unknown, the zero in Joseph's context represents a "tied game." Joseph drew on the relativity of the negative integers, illustrating runs below the tied score (i.e., the unknown referent).

Many contexts implicitly use integers as relative number: up and down runs in a baseball game without a known score (Wessman-Enzinger & Mooney, 2014); increases and decreases in money in a piggy bank with an unknown amount of money (Ulrich, 2012); getting on and off a train with an unknown number of riders (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2010). As learners coordinate their understanding of relativity with the integers in contexts, they must first determine a relative position, which points out the need for learners coordinate what a relative zero is.

Herbst (1997) reflected on how translation is related to relativity. He wrote, "The statement of an addition of the number line involves the juxtaposition of two arrows, a relative position" (p. 38). Herbst's reference to this relative position is similar to referencing a relative zero, or a starting location that uses 0. Similarly, Marthe (1982) used a river problem for investigating the thinking and learning of integer addition and subtraction. In this problem, the positive integers represented moving upstream and the negative integers represented moving downstream. This upstream and downstream movement is relative to the initial starting point on the river. Wherever one starts at on the river, represents the zero. Exactly where one starts at this river is unknown; yet, everything is measured from this point. This is a relative zero. Because part of conceptualizing relativity requires using zero as an unknown reference (with an infinite number of possibilities), this may be challenging for learners.

Ulrich (2012) referred to this use of zero as an unspecified reference point. Similar to Joseph's story where we do not know the score of the tied game, Ulrich defined an unspecified reference point as being able to conceptualize changes without an actual quantity known. Ulrich highlighted that this ability to think about relativity and use an "unknown" reference point, like Joseph did in his story, impacts students later in mathematics. For example, unknown reference points are important when working with vectors and matrices in linear algebra. Although we use conceptions of relativity beyond making sense of directed number, we lack explicit explorations of how these types of conceptions develop early on with directed number. The next section provides an example how a Grade 5 student constructed use of relativity with directed number.

13.4.3 Illustration of Relativity and Directed Quantities: The Case of Jace

Figure 13.3 illustrates the work of a Grade 5 student, Jace, for the same number sentence type (i.e., $-a + \square = b$, where $|a| < |b|$; Murray, 1985) at two different points in a 12-week teaching experiment (Steffe & Thompson, 2000) focused on integer addition and subtraction. Jace produced the drawing for $-4 + \square = 10$ during his first individual open number sentence session with me and produced the drawing for $-6 + \square = 15$ during his second individual open number sentence session. In the first session, Jace created an empty number line with -4 on the left and 10 on the right. He then used three sets of distances, with varying direction: -4 to 0 (a distance of 4,

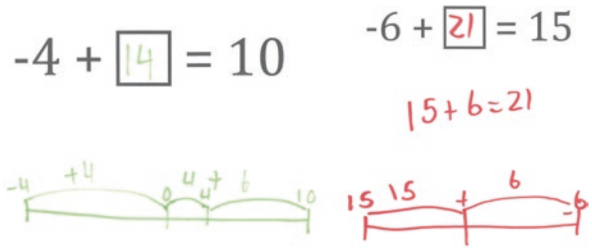


Fig. 13.3 Two number line drawings produced by Jace, a Grade 5 student, illustrating use of relativity with directed number

left to right), 0 to 4 (a distance of 4, left to right), and 10 to 4 (a distance 6, right to left). After summing his three distances, he concluded that the solution to -4 + □ = 10 is 14. In contrast, when solving -6 + □ = 15 2 weeks later, Jace placed -6 on the right side and 15 on the left of the empty number line. This differed from the first session in that the negative numbers are represented on the right side of the number line, rather than the left side. Also, he found two distances, instead of three: first examining 15 to 0 (a distance of 15, left to right) and then going from -6 to 0 (a distance of 6, right to left).

First, Jace's drawings of empty number lines highlight that the use of integers as a directed number ultimately requires using integers as relative number. Whether the integers are on the left or the right side of the number line, both of these representations are correct. Although our culture, curricula, representations of mental models, models in mathematics education, and research place negative numbers on the left side of a horizontal number line or on the bottom of a vertical number line, children do not necessarily attend to these conventions.

Second, Jace started to develop conceptions of zero as a referent. In the first drawing, he produced a number line with zero as a number on the number line. By the second drawing, we see that he, in fact, omitted an explicit 0. Yet, in both cases, he drew on 0 as a flexible referent to find the distances.

Third, Jace's strategy of finding the distance implicitly used directed number relatively for determining the absolute value or distance. Jace physically illustrated directed number from one relative number to another with motions and drawings, using these motions flexibly (sometimes right to left, other times left to right). When solving -6 + □ = 15, for example, Jace first moved his marker from left to right (i.e., 15 to 0) and then right to left (i.e., -6 to 0). Jace accounts for incremented distances verbally and flexibly, writes only these distances above his number line, and uses these distances to determine the solution of the directed distance (i.e., he sums of the two absolute values required to translate -6 to 15).

13.4.4 Connecting Themes of Relativity Across Mathematics Education and Developmental Psychology

Although the use of integers as relative numbers seems underemphasized in both mathematics education (i.e., not mentioned in curricula or standards documents) and developmental psychology (i.e., relativity has not been directly investigated), relativity is addressed both explicitly and implicitly in mathematics education and developmental psychology research. As illustrated above, the relativity of integers is an integral part of mathematics and thinking about directed number. What entails the starting point of the physical movement (i.e., our unknown referent)? Are the positive integers on the right or left side of a horizontal number line? Is the movement associated with adding or subtracting one-dimensional, or is that relative as well? These notions of relativity have implications for embodied cognition (e.g., Lakoff & Núñez, 2000), mental models (Bofferding, 2014), and integer instructional models (e.g., Saxe, Diakow, & Gearhart, 2013). Not only are integers relative numbers (Gallardo, 2002; Schwarz et al., 1993), conceptualizing relativity is implicitly imbedded in our work. Now, we need to learn more about how learners construct conceptualizations about relativity as it pertains to directed number.

13.5 Coordinating Translation and Integers as Directed Quantities

In contrast to the relativity of integers, coordinating translation and the integers as directed quantities represents a prominent theme in both mathematics education and developmental psychology from describing thinking to describing integer instructional models. In terms of integer instructional models, Herbst (1997) discussed the use of the number line metaphor as a way to make sense of integer addition and subtraction. Lakoff and Núñez's (2000) identification of order as a foundational component of mathematical cognition supports and informs the use of integers with translation. Whether talking about integer instructional models or ways of thinking about directed number, negative numbers may be constructed as point locations within this motion metaphor. Using a motion metaphor draws on the idea of symmetry on the number line (Herbst, 1997; Lakoff & Núñez, 2000). Ubiquitous pedagogical approaches support thinking about the addition or subtraction of integers as translations (e.g., Nurnberger-Haag, 2007; Tillema, 2012).

Although number lines (e.g., Saxe et al., 2013) and movement on linear scales (e.g., Nurnberger-Haag, 2007) are prevalent pedagogical tools, children do not necessarily construct movement or use number lines like top-down integer instructional models dictate (Wessman-Enzinger, 2018b). Rather, children create unique uses of movement and number line. These learner-generated constructions provide a conceptual tool for making sense of integers as directed number.

13.5.1 A Historical Perspective of Translation

The concept of a number line is foundational, not only for informing thinking and learning about translation, but also for informing current research in mathematics education on student thinking about number, and specifically negative integers (e.g., Bofferding, 2014; Saxe et al., 2013). Although historical developments of a concept, such as number line, may not parallel educational and psychological developments, a deep understanding of the past can offer researchers and educators perspectives on the present and help them make decisions for the future. As Sfard (2008) pointed out, "one becomes … bewildered when one notices the strange similarity between children's misconceptions and the early historical versions of the concepts" (p. 17).

Historically, we know that although some mathematicians had conceived of the number line in the seventeenth and eighteenth centuries (e.g., Wallis, 1685), most mathematicians and educators did not refer to number lines when attempting to make sense of operations with negative integers (Heeffer, 2011). Rather, mathematicians during the seventeenth and eighteenth centuries often made sense of negative integers by using contexts, such as debts of money, or incorporated geometrical approaches within explanations of rules of operations with negative integers (Wessman-Enzinger, 2018a). Heeffer (2011) presented historical evidence that mathematicians struggled in the past using number lines with operations, such as division, in their efforts to make sense of negative numbers and their operations. Indeed, the number line as a pedagogical tool evolved over several centuries to be incorporated into school mathematics (Wessman-Enzinger, 2018a)—with illustrations of the number line itself delayed for centuries after verbal descriptions of it. And, texts that included references to number line often paired it with contextual situations.

The historical struggle of mathematicians connecting operations with integers to the number line points to conceptual struggles of using the integers and number line; however, these are not necessarily places where the number line actually breaks down as contemporary learners engage with integers. Reflecting on potential breaking points for integers and number line, Liebeck (1990) stated, "The number line, then, emphasizes ordinality at the expense of cardinality" (p. 237). Liebeck hinted at the idea that the number line is not an infallible tool and certain integer instructional models, like number lines, offer different affordances. The number line is an important pedagogical tool, but specific tools may support some ways of reasoning more than others. Liebeck points to a conceptual leap that a child may have to undertake to begin to use the number line with integer operations—ordinality over cardinality.

In terms of using a number line, integer operations are often paired contrived rules. For example, Nicodemus (1993) described a "Linesman" where a human is standing on a number line facing right, negative number represents facing the opposite direction or walking backwards, and addition and subtraction represent moving forwards or backwards. Herbst (1997) also found these types of rules in a textbook analysis. For example, when considering the number sentence 2 − -3, it is suggested that one conceptualizes starting at 2 on the number line, turning around, and walking backwards three spaces on the number line, ending at 5.

These types of rules may not be intuitive to children, yet metaphors of movement are (e.g., Lakoff & Núñez, 2000). However, even these intuitions of movement do not guarantee that children will construct our cultural convention of a number line and translations on that number line. How will children use number line and integer operations, without us imposing our conventions and models on them?

We know that a major challenge that children may have, for example, with the number line is that the distance unit between the tick marks is to be used, not the tick marks themselves (Carr & Katterns, 1984; Ulrich, 2012). When learners count the tick marks, rather than the distances between the tick marks, they will end up with one more (or one less) than anticipated (Barrett et al., 2012). A major assumption with the number line as an integer instructional model is that learners will be able to extend their previous knowledge about whole numbers and the number line to operations with the integers and the number line. Ernest (1985) stated that the "number line model does not have any compelling inner logic. Instead it assumes familiarity with underlying representational conventions, which are to some extent arbitrary" (p. 418). Major assumptions of using the number line as an integer instructional model are that it is used in similar ways and that it supports learners' ways of thinking. Yet, we know that the same integer instructional model is not used in the same way by teachers (e.g., Murray, 2018). We also know that children create unique and sophisticated ways of working with integers (e.g., Bofferding, 2014; Bishop et al. 2016) that often surprise us.

Although the number line can certainly be tool for extending whole number reasoning with integers, we have to re-evaluate ways that it is developed and used. It took centuries for mathematicians to develop and use the number line; our students need time to develop use of the number line, particularly with integers. Learners may extend their use of a number line with whole number by using a number path (Bofferding & Farmer, 2018) incorporating negative integers. They may or may not use the number line as mathematically or culturally expected (e.g., Wessman-Enzinger & Bofferding, 2014; Wessman-Enzinger, 2018b). We cannot expect learners to create number lines that necessarily align with our cultural conventions.

13.5.2 A Contextual Perspective of Translation

Most research literature that discusses *transformations* of integers is specifically focused on translations (Marthe, 1979; Thompson & Dreyfus, 1988; Vergnaud, 1982) in contextualized situations for addition and subtraction only. While some researchers have pointed to using translation as a way to think about integer addition and subtraction (e.g., Wheeler, Nesher, Bell, and Gattegno, 1981), other researchers, like Marthe (1979) and Vergnaud (1982), have provided problem types that support translation as well. Bell (1982), Marthe (1979), and Vergnaud (1982) presented integer addition and subtraction as beginning with a relative number or initial starting point, using a translation, and then ending at a relative number or final ending point. Supporting this work, Bishop, Lamb, Philipp, Whitacre, and Schappelle (2014) shared that the children in their study solved integer problems with translation: "Starting point + Change = Ending Point." Directed number can be conceptualized as more than just "Starting point + Change = Ending Point," but also can be used with distances or difference (Bofferding & Wessman-Enzinger, 2017; Selter, Prediger, Nührenbörger, & Hußmann, 2012; Whitacre, Schoen, Champagne, &

Goddard, 2016). Thus, a variety of both contexts and problem types provide different opportunities for conceptualizing the integers (Wessman-Enzinger & Mooney, 2014; Wessman-Enzinger & Tobias, 2015).

Although many of the contexts used for whole numbers include discrete objects without movement (Carpenter, Fennema, Franke, Levi, & Empson, 2015), learners also engage in contexts with linear movement that supports translation as it begins to work with negative integers. We also know that different problem types support different ways of reasoning for whole number (Carpenter et al., 2015); this is likely the case for integers and translation as well.

For translation problem types, Marthe (1979) classified different problem types for additive structures for integers. The first category was S_iTS_f, where the initial state (S_i) is translated (T) to the final state (S_f). Marthe then described that any of S_i, T, or S_f could be the unknowns in a given problem. A second category Marthe described was $T_1T_2T_3$. He described T_1, T_2, and T_3 as "transformations" although they can also be described as linear translations. From this problem type, Marthe described that there are three subsequent problems that can be posed, where T_1, T_2, or T_3 are unknowns, and T_1, T_2, or T_3 have differing magnitudes and signs. Marthe provided contextual examples of each of these problems. For example, for the problem type $T_1T_2T_3$ with T_2 unknown, T_1 and T_3 with opposite signs, and $|T_1| < |T_3|$, Marthe provided the example, "A car makes an initial journey of 20 km upstream. Then it makes a second journey. If it had made only one journey from its starting point to its destination, it would have made a journey of 25 km downstream. Describe the second journey" (p. 156). Marthe stated that this problem type is more challenging than STS.

Temperature is an example of a context for connecting integer operations to directed number, with both translation and relativity (Altiparmak & Özdoğan, 2010; Beatty, 2010; Bofferding & Farmer, 2018). Using the context of temperature, we modified the Marthe (1979) problem types to include a distinction between directed distance and undirected distance, with state-state-translation (SST) and state-state-distance (SSD), respectively (Wessman-Enzinger & Tobias, 2015). When a problem is posed with two given relative numbers and the translation is unknown, this is classified as an SST problem. Whereas, when a problem with two numbers and a distance, without a clearly distinguished direction, this is considered to be an SSD problem (see Table 13.2 for the distinction between SST and SSD). Consider the SST problem type posed by a prospective teacher: "It was 12° outside Wednesday. It was 17 below zero degrees Thursday. How much had the temperature dropped since Wednesday?" Compare this to the SSD problem type posed by a prospective teacher: "One day in New York it is -14 degrees out. In Maine the same day it was -20°. What is the difference between the two states' temperatures?" The distinguishing feature of the SSD problem type from the SST is that no direction is provided in the problem. The problem types modified from Marthe (1979) are summarized in Table 13.2 below.

Similarly, in terms of the STS problem type, Vergnaud (1982) pointed out that the minus sign can illustrate a translation, or the minus sign can represent the inversion of a directed translation, which is more challenging. The "minus sign" is used

Table 13.2 Relativity and translation problem types from Wessman-Enzinger and Tobias (2015)

Problem type	Description
STS	A problem posed with a relative number and a translation, with the second relative number as the unknown
TTT	A problem posed with two given translations and the third translation is unknown
SST	A problem posed with two given relative numbers and the translation is unknown
SSD	A problem posed with two relative numbers and a distance, without specified direction

for finding differences; yet, the plus sign can also mean a difference between two directed numbers of different signs. Vergnaud, for example, provided "$x + (+4) = -3$, $x = (-3) - (+4) = -(3 + 4)$" (p. 73) and stated, "My view is that equalities and equations do not fit equally well all situations met and handled by learners, but only a few of them" (p. 74). In terms of using translation, Vergnaud made an important distinction that thinking about moving backwards two units from 1 may be represented by both the expression $1 - 2$ or $1 + -2$; however, these expressions may not *conceptually* represent this situation equally for the student.

As Vergnaud highlights, I have similarly found in my own work that children's thinking about contextual problems with integers, and the number sentences they write, do not always match the context (Wessman-Enzinger, in press). Three Grade 5 children (Alice, Jace, Kim) solved the following problem:

> The warmest recorded temperature of the North Pole is about 5° Celsius. The warmest recorded temperature of the South Pole is about -9° Celsius. Which place has the warmest recorded temperature? And, how much warmer is it?

Alice, Jace, and Kim each wrote different number sentences: $5 - -9 = 14$, $-9 + 5 = 4$, and $5 + 9 = 14$, respectively (Fig. 13.4).

How learners conceptualize this problem does not necessarily coordinate with the problem type, but it might. Here we see that both Alice and Kim obtained the solution of 14, yet Kim did not even use subtraction (i.e., $9 + 5 = 14$). For Kim, $5 - -9 = 14$ did not conceptually match this context; she stated that she did not agree that subtraction should be involved when one is adding distances. Similarly, Alice did not agree with $9 + 5 = 14$ initially because she stated that +9 is not in the context of the problem; -9 is. Alice's conceptualization matched the problem type (SST); but, Kim's conceptualization of the problem did not. In this vein, although various problem types for integers may provide insight into how learners solve problems, they do not necessarily solve the problems with translations as we expect.

Thompson and Dreyfus (1988) provided a rich instructional context in a microworld, called INTEGERS, for two Grade 6 students in order to investigate conceptions about integers. Within the microworld, the Grade 6 students solved contextual problems that were often of the problem type TTT, even illustrating directed numbers as linear vectors on a horizontal number line. For example, they constructed two different translations of a turtle and determined the net translation of the turtle with the vectors. Thompson and Dreyfus conducted the teaching experiment using

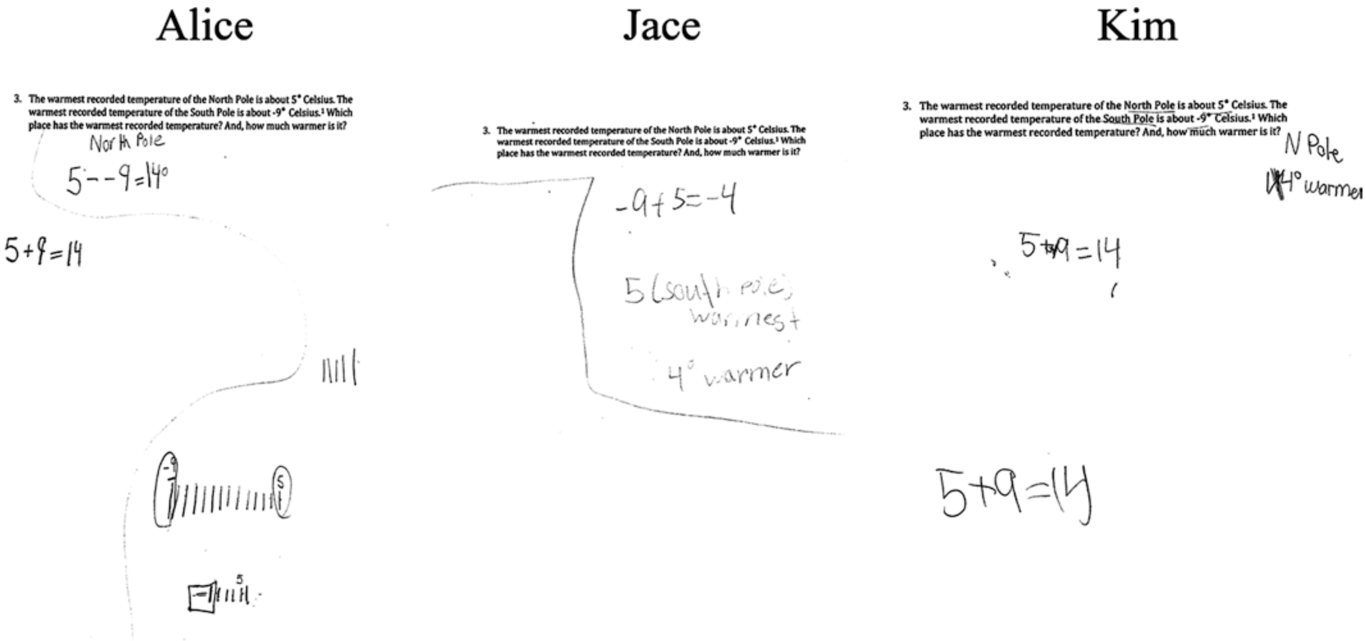


Fig. 13.4 Alice, Jace, and Kim's written work for the North Pole Problem

these net translations for 6 weeks. Similar to Thompson and Dreyfus (1988), but using a person instead of turtle, Liebeck (1990) included a number line with a person that moved along this number line. Liebeck's activity differed from Thompson and Dreyfus as it did not incorporate visualized directed vectors. Liebeck's activity also supported a different problem type—STS. Students in Liebeck's study started at different relative points, such as 2 or -5, translated the person from that point, and then found the ending point. Addition and subtraction of integers was described as "when we add, we move forwards" and "when we take away, we move backwards," respectively (p. 233). Liebeck provided a table for students to record the starting place, moving forwards or backwards, the ending place, and then the "answer" or the number sentence. This use of the person moving on the number line related to the STS problem. Both the contexts from Thompson and Dreyfus (1988) and from Liebeck (1990) used a conventional number line (e.g., partitioned, negatives on the left) and interpreted addition and subtraction unidirectionally (i.e., subtraction moves left on a conventional horizontal number line).

The contexts of Thompson and Dreyfus (1988) and Liebeck (1990) facilitated students' thinking about integers and translation, and there are many other contexts that may also support thinking about integers and translation. Some of these contexts include: a timeline with BC and AD dates (Gallardo, 2003); temperature increasing and decreasing (Wessman-Enzinger & Tobias, 2015); traveling up and down a river (Marthe, 1979); riding in an elevator (Iannone & Cockburn, 2006; Larsen & Saldanha, 2006); and balloons moving up and down (Janvier, 1985; Reeves & Webb, 2004). Despite all of these contexts supporting linear movement and directed number, some of the contexts support different types of conceptualizations of translation. The context used by Thompson and Dreyfus (1988) supports net translations (i.e., TTT); the context used by Liebeck (1990) and grounding metaphors with movement support identifying a relative number and translating to another relative number (i.e., STS); and, other contexts, like temperature (e.g., Wessman-Enzinger, in press), support using directed and undirected distance (e.g., SST, SSD). While there is often a quest for a "perfect" instructional model or a meaningful context for integers, these examples illustrate how working within a variety of contexts and problem types provides different opportunities to think about and work with integers as directed number, all of which are crucial for understanding integers.

Selter et al. (2012) differentiated between the *take-away* and *difference* models of subtraction. These models are related to both the problem types discussed above and to conceptualizations of translation. SST and SSD problem types are directly related to the difference model of subtraction, with one representing a directed distance and the other an undirected distance, and STS problem types seem related to the take-away model, with the change or "take-away" as a directed movement. Although STS, SST, and SSD are presented above as problem types in contexts, these problem types also point to ways that learners may conceptualize translations with integer addition and subtraction. Interpreting integer subtraction often requires a transition from take-away models of subtraction to distance models of subtraction (Bofferding & Wessman-Enzinger, 2017; Whitacre, Schoen, Champagne, & Goddard, 2016). Yet, our top-down instructional models for integer addition and subtraction do not explicitly support these transitions. Learners are capable of inventing their own constructions, using their own conceptions of translation to create ways to deal with integer subtraction and transition. Learners' ways of reasoning can be supported, in alignment to structures we understand (e.g., distance models, SST, SSD), without top-down integer instructional models (e.g., walk backwards, turn around, on a number line).

13.5.3 *Illustration of Translation and Directed Quantities: The Case of Kim*

The following example illustrates a student construction that differs substantially from conventional integer instructional models that support translation and directed quantities. Figure 13.5 illustrates a drawing of an empty number line from a Grade 5 student, Kim, at the end of a 12-week teaching experiment. Kim's number line highlights how she used directed quantity in unconventional ways. It is important to note that in this teaching experiment I did not provide any instructional models (e.g., chips models, number line models) to the students, and provided students only open number sentences or contexts without illustrations. The following excerpt of transcript is from when Kim solved $-1 - \square = 8$ shown in Fig. 13.5.

> Negative one minus something would equal eight. So if I did nine, it would be negative ten. So I turned it into a negative nine and so, it's sort of like this (starts drawing a number line). Here's a negative one (marks negative one on the number line), here's zero (marks zero on the number line). That's really far. Then right here is eight (marks eights on the number line). Then, nine, that's, they're both negative so you're going to subtract regularly. So like five minus three, you are going to subtract regularly with positive numbers, but it's negative numbers this time. When you do subtract it nine is a lot greater than the starting off number. So, it's going to hit zero when it's lost one (mark number line). And, then there's eight remaining over and then you can just like go into the positive though (waves hand to the right). You know, keep going with your remaining eight and get eight.

Kim constructed empty number lines like the one in Fig. 13.5, where she used STS as a strategy for integer subtraction: she started at a relative number and translated right to a second relative number. Integer subtraction like this conventionally may be thought of as the distance between two states, where $-1 - \square = 8$ would be conceptualized as the distance between -1 and -9. Instead, Kim uniquely used motion and a directed number starting at -1 and translating "-9" units to 8. Kim stated, "it's going to hit zero when it's lost one," decomposing the -9 units to -1 and

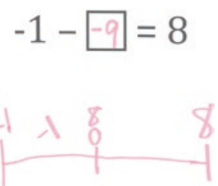


Fig. 13.5 Kim's unique use of translation and directed quantity with integer subtraction

-8 and using negative distance or directed number. Kim created a strategy where subtraction *moved right* (see Fig. 13.5), when traditionally subtraction involves moving left on this type of conventional number line (i.e., negative numbers on the left and positive numbers on the right). Comparing this to whole number reasoning, where addition moves right, marks this type of reasoning a powerful construction. Furthermore, she conceptualized distance as negative (see the "-1" written above the empty number line). Comparing the use of negative distance to whole number reasoning, where distance is always positive and not directional, marks another area of a distinct invention.

Kim's empty number line drawing and use of directed quantity highlight the uniqueness of her constructions, relative to typical instructional models for integer addition and subtraction. Kim's example provides evidence of a sophisticated and unique mathematical construction from a Grade 5 student. Her construction does not align well with current integer instructional models, yet does draw on the ideas of motion.

Using movement on the empty number line, Kim used her translations with addition and subtraction flexibly. That is, addition moved right on her empty number line (with positive directed distance) and subtraction also moved right on her empty number line (with negative directed distance). Comparing this to reasoning with whole number, where all directed distance is positive—addition moves right and subtraction moves left on a number line like hers—is novel. Furthermore, many instructional models for integer addition and subtraction maintain this type of whole number reasoning with the integer models (i.e., where addition moves right only and subtraction moves left only, but uses integer operations). Thus, Kim's construction and flexibility of using both addition and subtraction for moving right on her number line is powerful. Kim's construction offers a perspective on integer addition and subtraction where distance is relative (distance can be positive or negative) and movement is relative (subtraction can move right or left). Kim's invention highlights a way of thinking about integer subtraction absent from current integer instructional models and even subtraction models of take-away and distance.

13.5.4 Connecting Themes of Translation Across Mathematics Education and Developmental Psychology

Although the use of integers with translation is emphasized in both mathematics education and developmental psychology (e.g., use of movement on number line), the use of translation that is represented, both explicitly and implicitly, may be different from learners' constructions. As illustrated above, Kim used translation in an unconventional way. Her interpretation of subtraction with movement to the right marks a unique construction. She uniquely "lost" negative distance. What are other ways that children may create and construct translation? How are these unique constructions related to conceptions of relativity? For instance, if distance is interpreted as positive, then subtraction may be interpreted as a translation to the left (on a conventional number line). And, if distance is interpreted negative, subtraction may be interpreted as movement to the right (or left). Learning more about the depth of learners' constructions of translation and how this is related to conceptions of relativity has implications for embodied cognition (e.g, Lakoff & Núñez, 2000), mental models (Bofferding, 2014), and integer instructional models (e.g., Saxe et al., 2013), as it impacts the ways we leverage learning.

13.6 Concluding Remarks

The examples from students discussed here are intended to highlight and extend key themes in the literature: children are capable of creating robust and sophisticated constructions of translation and relativity in relation to integers as directed quantities, but we need to explore these constructions more in depth. Additionally, the examples are intended to challenge typical notions of what instructional models for integers entail. We must abandon the search for the holy grail of integer research—the illusive, infallible integer instructional model. Instead, let us take up pursuit of learners' robust and sophisticated constructions of integer operations.

Rather than using integer instructional models from top-down perspectives (instructional models created by teachers and researchers), we can draw on learners' constructions as the instructional models. As we look more towards learners' constructions, we should focus on overlooked ideas of relativity, paired with translation, for insight into directed quantity. Children have produced mathematical ideas (such as relativity) that have been overlooked in our own integer work. Yet, the ideas that the children have constructed are essential to directed quantity. As we learn more about conceptualizations of translation and relativity in relation to directed quantity, we can investigate how to leverage these student-constructed ideas to other advanced mathematics.

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