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Infinite Cardinalities, Measuring Knowledge, and Probabilities in Fine-Tuning Arguments (Chapter 5 of Knowledge, Belief, and God: New Insights in Religious Epistemology)

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Abstract

This chapter deals with two different problems in which infinity plays a central role. It first responds to a claim that infinity renders counting knowledge-level beliefs an infeasible approach to measuring and comparing how much we know. There are two methods of comparing sizes of infinite sets, using the one-to-one correspondence principle or the subset principle, and it argues that we should use the subset principle for measuring knowledge. The chapter then turns to the normalizability and coarse tuning objections to fine-tuning arguments for the existence of God or a multiverse. These objections center on the difficulty of talking about the epistemic probability of a physical constant falling within a finite life-permitting range when the possible range of that constant is infinite. Applying the lessons learned regarding infinity and the measurement of knowledge, the chapter hopes to blunt much of the force of these objections to fine-tuning arguments.

Keywords: fine-tuning arguments, infinity, measuring knowledge, multiverse, existence of God

Subject: Philosophy of Religion

Measuring Knowledge

One central element of being an expert involves having significantly more knowledge about a topic than the average person.[1] This seems an essential feature of expertise, since if an expert knew less (or roughly the same amount) about their field than an average person, we may wonder why we should consult them in the first place. We also commonly make comparisons between experts regarding how much they know in a domain or subdomain. For example, Brian and Beth are both physicians, but as an oncologist, Brian knows far more about cancer than Beth, a pulmonologist. And it may seem fairly plausible that having more knowledge must involve having more true beliefs.

But upon closer examination, two potential problems crop up: First, does it actually make sense to say that someone has *more* true beliefs than another person? Some philosophers find the ideas of counting, measuring, and individuating beliefs problematic. If they are right, it would make it quite difficult to say that Jones has more true beliefs than Smith. Second, even if we could coherently say that one person has more true beliefs than someone else, would this always be sufficient to improve one's epistemic status? If I have overwhelmingly many more true beliefs than you, but they all are about trivial and inconsequential matters, this does not place me in a better position epistemically, nor does it contribute to my expertise. Resolving these questions is important for understanding whether and how having more true beliefs or more knowledge contributes to having expertise. In this chapter, I will focus on an objection to the possibility of someone having more true beliefs than another person. I treat the problems of triviality and belief individuation elsewhere.

Nick Treanor's "The Measure of Knowledge" (2013) is the only extended discussion I am aware of that addresses the problem of measuring a person's knowledge. He notes that the claim that beliefs are not even theoretically countable is often made in the philosophical literature, though rarely supported by actual arguments (2013, 582). If such a claim regarding countability were right, it would fly in the face of the everyday intuitions and practices of non-philosophers when they attempt to compare how much people know, for example, subjecting students and professionals to academic and licensing examinations. Treanor illustrates the claim with quotes from philosophers such as W. V. O. Quine, Donald Davidson, and Barry Stroud, and he attempts to spell out why measuring knowledge should be considered problematic.

The most straightforward and natural way to measure how much S knows is to count how many beliefs S has that are true and possess the properties one considers necessary and sufficient for knowledge. When comparing people, the person who has the larger number of such qualifying beliefs is the one who knows more.[2] Treanor calls this view the cardinality approach, and he presents several difficulties for it. The difficulty I will focus on starts by considering the possibility of subjects who have denumerably infinite true beliefs (2013, 580). Some theories of mental representation entail humans having an infinite number of beliefs. If such a theory turns out to be true,[3] then since the cardinality of every countable infinity is the same, $\aleph_0$, every person's set of true beliefs would have the very same cardinality, even if A had, say, a million true beliefs that B did not have, while A had all of B's true beliefs. Treanor says this point is "decisive" if we switch to talking about ignorance (2013, 581), offering a reductio that would apply even to beings that have only finitely many beliefs. Suppose that cardinality is the measure of ignorance. Since there are infinitely many truths,[4] no matter how ↳ many truths I come to believe, I am still ignorant of an infinite number of other truths (as long as I do not know all truths); the cardinality of how many truths any non-omniscient being is ignorant of is precisely the same. But I clearly am not as ignorant as my ten-year-old self; therefore, cardinality cannot be the measure of my ignorance. Treanor treats ignorance and knowledge as inversely related—knowing more entails being less ignorant—and so if cardinality cannot be the measure of my ignorance, then it cannot be the measure of my knowledge either.

The correspondence principle and its consequences

Treanor's argument relies on Georg Cantor's one-to-one correspondence principle regarding the cardinality of transfinite sets. This principle states that as long as there exists a one-to-one mapping between all the elements of two sets, they have an equal cardinality, and it is responsible for some of the very counterintuitive consequences of his set theory, consequences that made some of his mathematical contemporaries reject it. Most applicable here is the fact that Cantor's transfinite set theory results in concepts of cardinality, equal number, and greater than and less than that run contrary in certain ways to what we usually mean by them.

A number of these counterintuitive consequences of the correspondence principle were pointed out before Cantor by a long tradition of Jewish, Christian, and Muslim philosophers arguing against actual infinites. If actual infinites were impossible, this would support the Abrahamic faiths' teaching that the universe was created and not infinitely old, and it would contribute as a premise to one type of cosmological argument for the existence of God. The earliest known member of this tradition was the Christian philosopher John Philoponus of Alexandria, who published his *de Aeternitate Mundi contra Proclum* in 529. Philoponus had previously argued against Aristotle's views on time and infinity, and in this work he argued for a beginning to the universe (Sorabji 1983, 198).

One class of arguments against actual infinities, *widening gap* objections, originates with Philoponus, and they involve traversals of an actual infinite at different rates. A good example is William Lane Craig's version of an argument of the medieval Muslim philosopher al-Ghāzāli: In the Ptolemaic system, for each revolution of the fixed stars, thousands of revolutions of Jupiter occur. So the gap between the total number of revolutions of Jupiter and the total number of revolutions of the fixed stars gets ever wider as time passes and approaches infinity. But after an infinite past, the number of revolutions Jupiter has made is equal in cardinality (in Cantor's sense) to the number of revolutions the fixed stars have made, resulting in no gap at all between the two, precisely the opposite of what we would expect (Craig 1979, 98).

Richard Sorabji describes a different widening-gap conundrum, courtesy of Nicholas Denyer. We need to explain "the sense in which a man who has spent 364 days of every past year in hell has spent more time there than the man who has spent one day of every past year in hell" (Sorabji 1983, 218). Both can say that they have been in hell an infinite number of days, but should not the first man be able to say that he has

suffered many more days in hell than the second? Sorabji points out that "however large a *finite* period we take, the ratio of days in hell remains at 364:1" (italics in original). When we consider all of an infinite past, however, the correspondence principle tells us they have spent the same number of days in hell. This is true no matter how lopsided we make the ratio; we can change the story so that for every billion-year period in the past, the first man spends every day of each billion-year epoch in hell except for the very last day of each epoch, when the second man takes his place for just that day. Yet it seems clear that the first man has suffered far more, and if God were to ask us to choose to somehow take on the entire experience of one of these men from eternity past, there would be no doubt: we would immediately choose the second.

Widening-gap objections rely on the intuition that the behavior of infinite sets should be similar to the limit case of finite sets as they tend toward infinity. So, for example, in case of the two men and hell, the difference between the days they have spent in hell can be expressed as $(364n - n)$, where n is the number of years that have passed:

$$\lim_{n \to \infty} (364n - n)$$

As n tends towards infinity, the difference, $(364n - n)$, should also approach infinity. But then at infinity, the difference between the total numbers of days suddenly disappears. This expression's behavior differs radically between a potential infinite (the limit case) and an actual infinite, and this sharp disparity presents a seemingly paradoxical result of applying the correspondence principle.

A different class of objections made by these philosophers involves adding to and subtracting from infinite sets. Addition's oddness is brought out by Hilbert's hotel, which has an infinite number of rooms, all of which are occupied. If a new guest comes and asks for a room, the manager can shift all the guests up one room, 1 to 2, 2 to 3, and so on, and then put the new guest in room 1. Even an infinite number of additional guests can easily be accommodated by the hotel, by shifting all the current guests from room n to room $2n$. And this addition of an infinite number of additional guests can be repeated again and again to infinity without any change in the cardinality of the number of guests residing in the hotel. This is clearly contrary to our normal concept of addition, but at least it is self-consistent.

Craig argues that subtraction, on the other hand, gives us inconsistent results: Imagine a library with an infinite number of books, with each book labeled with a natural number. Suppose we take out every other book. We have removed an infinite number of books, and the number of books left in the collection remains the same, ↳ since there is a one-to-one mapping between the remaining books and the original set. In fact, we could repeat this removal procedure an infinite number of times and there would be the same number of books remaining.[5] Yet if we start over with the original library and remove every book with a number greater than 3—which would be taking out the same infinite number of books as our original removal—then the infinite library suddenly becomes a finite one (Craig 1979, 86).

This problem with subtracting from infinite sets undermines Treanor's reductio about our ignorance of infinite truths. Recall that his argument goes something like this:

> (T1) Suppose that cardinality is the measure of my ignorance.
>
> (T2) No matter how many truths I come to believe, I am still ignorant of an infinite number of other truths given my not possessing omniscience; the cardinality of how many truths I am ignorant of remains the same.
>
> (T3) But I am less ignorant than my ten-year-old self.
>
> (T4) Therefore, cardinality cannot be the measure of my ignorance.
>
> (T5) Knowing more entails being less ignorant.
>
> (T6) Therefore, cardinality cannot be the measure of my knowledge.

We can construct a parallel argument:

> (B1) Suppose cardinality is the measure of the number of books that have not been written by the human race.

> (B2) No matter how many books we write, an infinite number of other unwritten books remain; the cardinality of unwritten books remains the same.
>
> (B3) But there are far fewer unwritten books now than there were before the rise of writing, with particularly many written in recent centuries.[6]
>
> (B4) Therefore, cardinality cannot be the measure of unwritten books.
>
> (B5) Having more written books entails there being fewer unwritten books.
>
> (B6) Therefore, cardinality cannot be the measure of written books.

This conclusion, however, contradicts the fact that the proper metric of written books *actually is* counting how many books have been written.[7]

There are at least three problems with Treanor's argument, the first two involving subtraction from infinities: First, he removes all the truths we have come to believe since an earlier time (possibly an infinite number of truths for certain kinds of minds) from the infinite set of truths we were ignorant of at an earlier time. Second, to interconvert between ignorance and knowledge in the move from T4 to T6, he has to assume that he can get the number of beliefs we know by subtracting from the total number of truths the number of truths we are ignorant of, the latter two sets both being infinite. Since subtraction in Cantorian set theory gives us inconsistent[8] results, both moves are illegitimate.

Finally, and most importantly, Treanor's reductio does not actually give him what he wants: T4 does not logically follow from the premises. Instead, if T3 is true, what we should actually conclude from it should be the disjunction cardinality is not the measure of our ignorance (¬T1) *or* the cardinality of how many truths I am ignorant of does not remain the same as I come to believe more truths (¬T2). If one assumes the Cantorian definition of cardinality, which relies on the correspondence principle, as being the correct one to use when dealing with infinite sets, then the second disjunct is false and the conclusion should be the first disjunct. But if we had a different definition of cardinality for infinite sets than Cantor's, one that allows for different sizes of denumerably infinite sets, then the second disjunct could be true: the number of truths I am ignorant of would decrease as I learned, even while remaining infinite. Many mathematicians and philosophers assume with Treanor that the Cantorian definition of cardinality is the right one or even the only one available, but there is a history of thinkers (including Cantor himself in a way)[9] who have recognized that there are actually several ways that we can define the concepts of number and cardinality for infinite sets.

The subset principle

The major rival to the correspondence principle is the subset principle, which states that a set that is a proper subset of another set is smaller in size. While both principles always agree in their judgments regarding relative size when comparing finite sets, they can disagree when it comes to pairs of infinite sets. This can be illustrated by another class of counterintuitive consequences of Cantorian cardinality: subset paradoxes. Galileo said that it seems obvious that there are far fewer square numbers (1, 4, 9, 16, ...) than there are natural numbers, since the square numbers are all natural numbers, forming a proper subset of the natural numbers, and there are many non-square numbers among the natural numbers. Yet he also pointed out that the two groups of numbers can be paired off, each square number with its square root, and so the two groups are of the same size (Galileo [1638] 1954, 32, cited in Parker, 88). Interestingly, Galileo throws in a widening-gap argument for good measure—the ratio of square numbers to natural numbers decreases as we get to larger sets of numbers:

> Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10,000, we find only 1/100 part to be squares; and up to a million only 1/1,000 part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.(Galileo [1638] 1954, 32)

Though Galileo does not speak in these terms, once again, there exists a huge disparity between the behavior of the limit and the infinite.

So the square numbers and natural numbers are of equal size according to the correspondence principle, while the subset principle says that the set of square numbers is smaller. Which principle should we adopt as the correct principle for comparing infinite sets? We have several options. Matthew Parker notes that Galileo anticipates Kant in rejecting both principles (2009, 92), saying that:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. (Galileo 1954, 31)

For Galileo, we simply cannot apply our concepts of relative size to infinite sets. So one option would be to remain silent on comparisons of such sets. A pair of options involves dropping one of the principles and designating the other as the one true principle: Cantor obviously held onto the correspondence principle, while Bernard Bolzano thought the subset principle was the correct one (Moore 2001, 113; Parker 2009, 94–5). Cantor's overall approach turned out to be far more mathematically ↳ fruitful and useful than Bolzano's, and so his choice between these two intuitions has been widely assumed. Nevertheless, as we saw above, there are several consequences of this choice that seem paradoxical and go against strong intuitions we have concerning size. A fourth, "more pluralistic" approach is suggested by Parker, who partially agrees with Galileo in saying that our everyday concept of numerosity does not apply to infinite sets, but he also argues that we can define at least two different extensions of that concept, each utilizing one of the principles of set size. Neither is "the uniquely *right* concept," but each is useful in different ways (2009, 91, Parker's emphasis).

This last approach seems correct. Since our everyday concepts concerning relative size cannot be used consistently with infinite sets, we need to decide which of the possible extensions of those concepts is most suitable and useful with regard to each particular question or domain. So which extended concept should we use with regard to comparing how much knowledge agents possess? To answer this question, let us consider what we want to express when we say that person A knows more than person B. What do we actually want when we ask whether someone knows more than another person? Is it that one could pair off their true beliefs to the other's true beliefs in a one-to-one correspondence? Clearly, it is the latter concept that interests us.

A few examples should make this even more obvious. Imagine that God has assigned names to each of the natural numbers. He then discloses to angel A the names of all the natural numbers, while angel B is given only the names of the even natural numbers. It is obvious that angel A knows far more than angel B, despite there being a one-to-one mapping between their two sets of beliefs. It does B little good to claim that he knows as much as A by appealing to the correspondence principle, since there are an infinite number of questions A can answer correctly (namely, the names of the odds) that B cannot, in addition to A knowing all the names B knows. If we were playing a game where winning involved knowing the names of randomly selected numbers, would we be indifferent as to which angel to choose for our team because the correspondence principle says they know the same number of names? Or would we clearly prefer angel A because A's knowledge is a proper superset of B's and we would be twice as successful in the game if we went with A?

We can sharpen this preference further by appealing to Cantor's proof that there is a one-to-one correspondence between the points on a finite line segment and all the points in an infinite three-dimensional space (or any *n*-dimensional space, for that matter), if we associate points with real numbers (Moore 2001, 118). If angel A knows the temperature at every point in a spatially infinite universe, while angel B only knows the temperature at every point on a 1mm long line segment situated, say, along one of the corners of the Empire State Building, it seems absurd and irrational to say that B knows as many truths as A. If winning a game involved knowing the temperature at a randomly chosen point in the universe, wouldn't the choice of which angel to be on our team make an enormous difference to our prospects of success?

So when we are thinking about how much subjects know, it seems clear that we want to extend the concepts of 'more,' 'less,' and 'equal' in the subset direction. This extension ↳ suits our purposes with regard to knowledge far better than the correspondence extension. Thus my ignorance does decrease in this sense when I learn new truths, despite the fact that I am still ignorant of an infinite number of truths, since what I am ignorant of now is a proper subset of what I was ignorant of before I came to acquire a truth previously

unknown to me. And minds that have infinite sets of true beliefs can differ in how many true beliefs they have.

Non-subset comparisons

Before turning to fine-tuning, let us look at a remaining problem. In a footnote, Treanor does briefly mention that proper subsets might give us a sense of knowing more or less despite equal infinite cardinalities, but he says that these kinds of cases would be "too rare to be helpful," as his ten-year-old self, while not knowing much of what he knows now, would know many things that his present self no longer knows (2013, 600, n. 5). Framing this objection in terms of Treanor's ignorance argument, while the infinite set of truths I am ignorant of now is possibly a proper subset of what I was ignorant of, say, ten minutes ago, it is certainly not a proper subset of what I was ignorant of yesterday, since I have forgotten many trivial details of yesterday's activities in the intervening time.

Still, it seems intuitively possible to make judgments with regard to relative infinite sizes in certain kinds of cases where there are no proper subsets involved. Take, for example, a case similar to the two angels case above, where angel A* knows the names of all the natural numbers, but instead of knowing the names of the even natural numbers, B* knows only the names of the negative even numbers. In this new case, B*'s true beliefs do not form a subset of A*'s true beliefs, and yet it seems just as obvious that A* knows more than B* does.

Or consider the two men in hell scenario, where one man spends 364 days of each year in hell, while the other only one day per year. If we specify which days of each year that each man is scheduled to spend in hell—say, the first man spends every day in hell except for New Year's Days and leap days, and the second man spends only New Year's Days in hell—then we can generate two sets of days spent in hell that are not in the proper subset relation with each other. But it still seems obvious that one contains many more days than the other. We can adjust the story so that they *are* proper subsets of each other, by changing the second man's scheduled day to Christmas Days, but merely shifting the scheduled day shouldn't make a difference in the relative sizes of their sets of days spent in hell.

Finally, here is a geometric example. Imagine two line segments, L1, stretching from 0 to 1 along the *x*-axis, and L2, stretching from 0 to 2. Both line segments contain an ↳ infinite number of points, but from the subset extension perspective, L2 contains more points, since it includes every point L1 has and more besides. Now imagine a third line segment further down the *x*-axis, L1*, which stretches from 3 to 4. There is no overlap between L1* and L2, and yet it still seems obvious that L2 contains more points than L1*, since L1 and L1* share the same length. Another way to think about this example would be to imagine L1* as simply L1 moved over some distance. Sliding L1 over (while not transforming it in any other way) should make no difference in terms of how many points it contains. We can generalize this by noting that we are able to compare not only the lengths of lines, but also the areas of planes and the volumes of regions of space. It seems arbitrary and odd to be able to compare the infinite numbers of points within them (using the subset extension) if one region contains another and not be able to do so if they do not contain each other or overlap.

The problem arises, however, when we try to justify our intuitions here. In the modified angels case, one reason why we might think that B* has fewer beliefs is because B*'s beliefs regarding the names of the negative evens in the modified story match up one-to-one to B's beliefs about the names of the positive evens in the original scenario, where a proper subset relation does exist between A and B's sets of beliefs. But if we have dropped the correspondence principle in pursuing an extension of cardinality using just the subset principle, we cannot appeal to one-to-one correspondence to claim that the numbers of beliefs B and B* have are the same in both scenarios. Similarly, for the men in hell cases, one might be tempted to try to show equal size by pointing to a one-to-one correspondence between all the New Year's Days and all the Christmas Days.

Still, it may be possible to avoid appealing to the correspondence principle and define an extension of subset cardinality that can apply to sets that are not in the proper subset relation with each other. Not being a mathematician or set theorist, I can only offer a sketch of some thoughts that could begin to help in producing such an extension. When relative rates are involved, such as the men in hell cases, we can appeal to equal relative rates and limit case behavior to avoid using the correspondence principle. The second man in the hell cases accumulates days in hell at the same rate, one day per year, whether that day is Christmas

or New Year's. On the subset principle extension of cardinality, limit case behavior correctly predicts whether the quantities are equal at the actual infinite and which quantity is larger if they are not equal (unlike on the correspondence principle extension). So since the limit behavior of the total number of days in hell is the same whichever day we schedule for the second man, we could argue that he suffers the same total number of days in both scenarios.

It is difficult to generalize this argument to situations where relative rates and limits are not already involved, such as the modified angels case. We may want to introduce a baseline from which to generate relative rates, as Galileo did when he pointed out the diminishing ratio of square numbers to natural numbers as we count up to infinity; in his case, the baseline happens to coincide with one of his sets, but a somewhat clearer ↳ example would be to use the natural numbers as a baseline to compare the square numbers with the cube numbers. Or we could use something else that varies together with the quantity we are interested in; in the geometric example, we used length (or area or volume) as a proxy measure for the number of points. But these approaches only work when there is such a co-varying measure or when the sets are already subsets of the baseline. For the modified angels case, since one knows the positive integers and the other knows the negative evens, they do not have a shared baseline for comparison, nor is there some other variable that we could use to compare their sizes. It seems necessary that we would have to appeal to a bijection function (utilizing one-to-one correspondence) in order to convert the negative evens into a corresponding set of positive evens or to convert the positive integers into a baseline of negative integers for the negative evens. So it seems that we can fully justify only certain kinds of infinite non-subset comparisons, by utilizing relative rates, limits, and measures when they are available.

Fine-Tuning and Coarse-Tuning

Another issue involving infinity, known as the normalizability problem or the problem of infinite ranges, is raised as an objection to the argument from cosmological fine-tuning to the existence of either God or a multiverse. Physicists in recent decades have discovered that for many constants of physical laws and initial conditions of the universe, if those constants or initial conditions were only very slightly different from what they actually are, life that utilized complex chemistry of any sort could not exist (see Collins 2009 for an overview of the scientific evidence and the argument). Proponents of the fine-tuning argument (FTA) take this set of discoveries as a premise for a probabilistic argument that says that given how extremely unlikely it would be that the constants and initial conditions be in such a tiny life-permitting region of values in a single atheistic universe, the facts of fine-tuning give evidential support for a God who intentionally set these constants so life could exist, or for a multiverse in which it was very likely that at least one universe with life-permitting constants would exist, or, combining these hypotheses, a theistic multiverse.

To arrive at the improbability of a constant being in that tiny life-permitting region, we need to have a range of that constant's possible values, with the life-friendly region constituting only a sliver of it. Any finite limit for this range of possible values would be arbitrary without some principled reason for such a limit, so it seems the comparison range should be infinite. However, infinite ranges cause trouble for probabilities, as the physicist Paul Davies points out:

> The problem is that there is no natural way to quantify the intrinsic improbability of the known "coincidences". From what range might the value of, say, the strength of the nuclear force (which fixes the position of the Hoyle resonances, for example) be selected? If the range is ↳ infinite, then any finite range of values might be considered to have zero probability of being selected. But then we should be equally surprised however weakly the requirements for life constrain those values. This is surely a *reductio ad absurdum* of the whole argument.
>
> (1992, 204–5)

Philosophers have elaborated on this problem. Neil Manson argues that if we accept an epistemic probability of 0 for any finite range to be selected from an infinite range, then it becomes too easy to get a good fine-tuning argument; we guarantee it "on the cheap":

> all one would need do is show that there is at least one cosmic parameter for which life constrains the possible values to a finite interval. Then one would have shown that the probability of a life-

> permitting universe is zero no matter how large that interval. Furthermore, there would be no need to find any further cases of fine-tuning2, for no additional evidence could make it any less likely on the chance hypothesis that the universe is such as to permit life.
>
> (Manson 2000, 347)

Yet many philosophers and cosmologists believe that having multiple fine-tuned parameters that are physically independent of each other *does* add further weight to the force of the fine-tuning argument.

Timothy McGrew, Lydia McGrew, and Eric Vestrup emphasize the non-normalizability of an infinite range:

> Probabilities make sense only if the sum of the logically possible disjoint alternatives adds up to one—if there is, to put the point more colloquially, some sense that attaches to the idea that the various possibilities can be put together to make up one hundred percent of the probability space. But if we carve an infinite space up into equal finite-sized regions, we have infinitely many of them; and if we try to assign them each some fixed positive probability, however small, the sum of these is infinite.
>
> (McGrew, McGrew, and Vestrup 2001, 1031)

If, on the other hand, we assign to each finite-sized region a probability of 0, there is no way we can get to 1 for the whole range, since the sum of even an infinite number of 0s is still 0. So there seems to be no way to use standard probability theory to deal with infinite ranges. What if we were to use limits, as I did in the earlier section on measuring knowledge, and extrapolate from how the probability behaves as the size of a finite comparison range approaches infinity? The limit of the probability would be 0, and the McGrews and Vestrup offer their own version of the reductio Davies and Manson suggest, coining the commonly used name for it.

> For using such reasoning we can also underwrite what we shall call the “Coarse-Tuning Argument”. Suppose … the various parameters, rather than being constrained to within tiny intervals around those that characterize our own universe, could take any values within a few billion orders of magnitude of our values. It is hard to imagine anyone’s being surprised at the existence of a life-friendly universe under such circumstances. Yet the “ball” in this case is isomorphic to the ball in the FTA: both of them have measure zero in R_+^K. In consequence, any inference we can draw from fine-tuning is not only paralleled by a coarse-tuning argument; it also has precisely the same probabilistic force … we are confronted with an unhappy conditional: ↳ if the FTA is a good argument, so is the CTA. And conversely, if the CTA is not a good argument, neither is the FTA.
>
> (McGrew, McGrew, and Vestrup 2001, 1032)

Surprisingly, several prominent defenders of the fine-tuning argument embrace the conclusion of what is intended to be a reductio. Robin Collins (2005, 403), Jeffrey Koperski (2005, 310–12), and Alexander Pruss (2005, 415, 421) each propose tweaked versions of probability theory that avoid the normalizability problem (for example, Pruss drops countable additivity) and say that the probability of any finite region with an infinite comparison range and even probability distribution is 0. And each of them bites the bullet and embraces the claim that the coarse-tuning argument is a good one and is just as strong as the fine-tuning argument. Collins says we “just have to conclude that our initial impressions were wrong that it was the smallness of the range, not its finiteness, that gave the FTA its force” (2005, 404).

This willingness to bite the bullet is puzzling, and I side with the McGrews when they argue that:

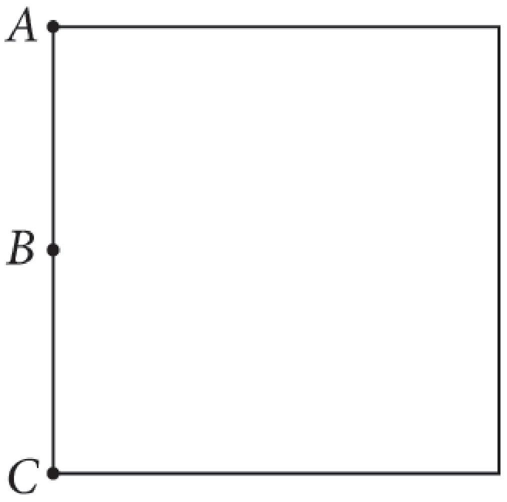
> *all* of the excitement about the FTA has centered on the alleged narrowness of the life-friendly regions … We doubt that anyone would have considered the CTA to be even a possible argument for design were it not for the objections that have been raised against the FTA.
>
> (McGrew and McGrew 2005, 433)

I want to emphasize the oddness of the claim that the CTA is not only a good argument, but is equal in strength to the FTA, or, for that matter, every other similar argument with a finite range, no matter how large its size. It seems obvious to me that if we were to discover further physical constants that required

independent fine-tuning for life, or if we found that the life-permitting ranges of the known fine-tuned constants and initial conditions are substantially narrower than physicists currently believe, the FTA would be substantially strengthened.

Even if the CTA proponents' non-standard probability theories say that the probability of an extremely narrow region is precisely the same as the probability of a much larger one, both 0, they are clearly *not* the same in all probabilistic respects. Suppose an infinitely sharp dart is thrown randomly at a square (see Figure 5.1).

Figure 5.1



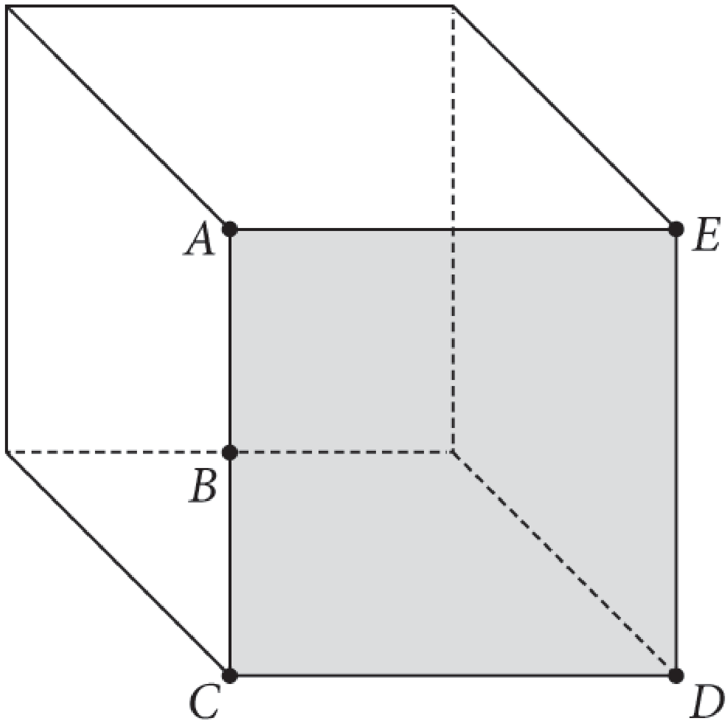
The probability of the dart hitting anywhere on the left edge of the square, *AC*, is 0, as is the probability of it hitting the upper half of that edge, *AB*. But there is a conditional ↳ probability of 0.5 that the dart hits the upper half of that edge (*AB*) given that the dart hits somewhere along the whole edge, *AC*.[11]

$$P(AC) = 0 \qquad P(AB) = 0 \qquad P(AB|AC) = 0.5$$

Even if the absolute probabilities of hitting *AC* (the left edge) and hitting *AB* (the top half of *AC*) both are 0, this does not mean there is no probabilistic difference between them. The dart is twice as likely to hit somewhere along the whole left edge as it is to hit the top half of the left edge. Both standard and CTA-friendly probability theories fail to capture this difference.

In fact, there are an infinite number of different levels of absolute probability 0. Add another dimension to the square we have been considering and suppose we randomly choose a point in the resulting cube (see Figure 5.2).

Figure 5.2



The probability of the point being anywhere on its shaded front face, *ACDE*, is 0. The probability of it being along one of that face's edges, *AC*, is also 0. Yet there is an even greater conditional probability disparity here than in the square case, since the probability of the point being on the edge *AC* conditional on the point being on the face *ACDE* is 0, not the non-zero conditional probability of the dart hitting *AB* on it hitting *AC* along the edge of the square:

$$\mathrm{P}(AC|ACDE) = 0 \qquad \mathrm{P}(AB|AC) = 0.5$$

So the randomly chosen point landing on the cube's face is infinitely more likely than it landing on the edge, and yet their absolute probabilities are still both 0. And we can iterate this procedure to higher and higher dimensional spaces to produce an infinite number of levels of absolute probability 0 that still have very different conditional probabilities.

Mark Colyvan, Jay Garfield, and Graham Priest also press normalizability objections against the FTA, but they point out an important fact:

> Admittedly, the problem here has nothing to do with the fine-tuning argument; it concerns (standard) probability theory. Indeed, similar reasoning shows that an infinitely sharp dart cannot hit a dart board, no matter how big the board is; or that no one can win a lottery that has infinitely many tickets. It's true that such paradoxes of probability theory are well known and that there does not seem to be any consensus on how they are to be resolved.
>
> (2005, 327)

Since this is a problem in probability theory that everyone needs solved, whatever the solution turns out to be for these standard paradoxes, it would likely help with the infinite ranges in the fine-tuning argument. After all, the infinite range problem is quite analogous to infinite lottery cases. There is clearly a difference between the probability of winning an infinite lottery where only two tickets are drawn as winners and a lottery where a googolplex (10^{googol} or $10^{10^{100}}$, a 1 with 10^{100} zeros after it) tickets are chosen. One's surprise at winning should be far greater in the two-winners lottery scenario. After all, there are half a googolplex times more winners in the googolplex lottery than there are in the two-winner lottery.

While I am not in a position to resolve the more general problem for probability theory in this chapter, I want to point to what I believe to be a major source of the problem as well as a promising direction to pursue. In the section on measuring knowledge, we saw that subtraction involving infinities is inconsistent. Similar problems arise when dividing by infinities. Consider these sets: the natural numbers, the even

natural numbers, and the set of natural numbers divisible by 10. All these sets have the same Cantorian cardinality, and so thinking about division one way, we might think that taking any one of these sets and dividing it by any other (or by itself) should always give us 1. But from another perspective, the subset perspective, dividing different pairs of these sets should give us different results. The positive evens divided by the natural numbers should give us 0.5, the set of natural numbers divisible by 10 divided by the natural numbers should give us 0.1, while the natural numbers divided by the evens should give us 2. This last example mirrors my case of angel A knowing the names of all the natural numbers, while angel B only knows the names of the evens; it is natural to say that A knows twice as many names as B does, and that the ratio of how much B knows to how much A knows (with regards to the names of numbers) is 1/2.

The question of how to divide with infinities is clearly relevant to the infinite range objection to the fine-tuning argument (as well as the infinite lottery paradox), and developing an alternative probability theory based on a subset extension instead of the standard correspondence one may be key to resolving these paradoxes of infinity. In fact, a group of mathematicians and philosophers have recently been working on a theory like this, which they call "numerosity theory" (Benci, Horsten, and Wenmackers 2013).[12] It is still early days for this theory, and to evaluate it goes beyond ↳ the scope of this chapter, but if it does succeed, it may provide the probabilistic foundations for a version of the fine-tuning argument that deals gracefully with infinite ranges and yet does not require that coarse-tuning arguments be just as strong as the FTA.

In the meantime, let me offer some considerations for why the CTA should not be considered to be just as strong as the FTA. Let w be the width of the wide life-permitting range R_W in the CTA, while n is the width of the narrow life-permitting range R_N of the FTA, and R_W includes all of R_N. Suppose n is several billion times smaller than w. For any finite comparison range larger than w, the ratio of the probability of the constant being in R_N to the probability of the constant being in R_W is n/w. As the size r of the comparison range grows towards infinity, the individual probabilities get increasingly closer to 0, but their ratio remains constant.

$$\lim_{r\to\infty} \mathrm{P}\left(\mathrm{R_N}\right) = 0 \qquad \lim_{r\to\infty} \mathrm{P}\left(\mathrm{R_W}\right) = 0 \qquad \lim_{r\to\infty} \frac{\mathrm{P}(\mathrm{R_N})}{\mathrm{P}(\mathrm{R_W})} = \frac{n}{w}$$

At infinity, we have a choice between extensions for how the ratio will behave, similar to the choice we had for limits in the previous section of the chapter.

We saw there that the subset extension agrees with limit behavior (at least in the sense of which of two quantities is greater or if they are equal), in contrast to the one-to-one correspondence extension, which allows for infinite behavior that is radically different from limit behavior. With the widening-gap cases (the revolutions of Jupiter vs. the fixed stars and the men in hell for differing numbers of days per year), the subset extension gave the correct answer. In the square dartboard case, the dart is twice as likely to hit somewhere along the whole left edge as it is to hit the top half of the left edge, even if the absolute probabilities for these two possibilities are both 0. Consider taking just the left edge as a line segment, getting rid of the rest of the square. The dart now always lands somewhere on that line segment. The absolute probabilities of a dart hitting somewhere on that line segment and it hitting its top half are 1 and 0.5. Moving from this finite case back to an infinite one by adding back the rest of the square does change the absolute probabilities back to 0, but the ratio, and hence the relative probabilities of hitting those line segments, remains the same. This seems to tell against a choice of extension at infinity that would lead to discontinuous behavior of the ratio.

What's more, we can run a conditional probability argument similar to the dart example when comparing a wide life-permitting range R_W with a narrow life-permitting range R_N. Even with an infinite comparison range, the conditional probability that a randomly chosen number lies in R_N given that it lies in the larger R_W, will be n/w. Since n is several billion times smaller than w, that conditional probability will be tiny.[13] If we ↳ go the other way, the conditional probability of being in R_W on a number having been randomly selected in R_N will always be 1 because of the superset relation the larger range has to the smaller one. This disparity in conditional probabilities will indicate which range has the larger relative probability and by how much, as long as there is a proper subset relation between the two finite ranges. Why? Because no matter what the absolute probabilities are, if the conditional probability of a number in R_N being picked given that a number was randomly chosen in R_W is tiny, then it is that much less likely for the number to fall in R_N than it is to fall somewhere in R_W.

Furthermore, we can compare the epistemic impacts of different possible pieces of evidence on hypotheses using only ratios of Bayesian likelihoods. The ratio of the widths of R_N and R_W, *n*/*w*, can then be used to show how big the ratios of likelihoods is between the evidence of fine-tuning and the evidence of coarse-tuning, and thus their relative evidential force. Let's start with the relative odds form of Bayes Theorem:

$$\frac{P(H_1|E)}{P(H_2|E)} = \frac{P(H_1)}{P(H_2)} \frac{P(E|H_1)}{P(E|H_2)}$$

The following are the ratios of the posterior probabilities of the disjunctive hypothesis theism or multiverse (TM) and the single naturalistic universe hypothesis (S) after conditionalizing on the discovery of coarse-tuning (C) compared with conditionalizing on the discovery of fine-tuning (F):

$$\frac{P(TM|C)}{P(S|C)} = \frac{P(TM)}{P(S)} \frac{P(C|TM)}{P(C|S)}$$

$$\frac{P(TM|F)}{P(S|F)} = \frac{P(TM)}{P(S)} \frac{P(F|TM)}{P(F|S)}$$

To see the relative evidential impact of coarse-tuning vs. fine-tuning, we divide these two ratios and get:

$$\frac{\frac{P(TM|C)}{P(S|C)}}{\frac{P(TM|F)}{P(S|F)}} = \frac{P(C|TM)}{P(F|TM)} \frac{P(F|S)}{P(C|S)}$$

What is the value of the second term P(F|S)/P(C|S)? If we calculate the constituent probabilities individually and then attempt to divide, both will be 0 and we cannot divide by 0. However, if we use the subset extension in light of my arguments regarding conditional probabilities and the limit behavior of the ratio of probabilities, we can avoid the division by 0 and directly conclude that P(F|S)/P(C|S) is *n*/*w*.

Since *n*/*w* is so extremely close to 0, the first term P(C|TM)/P(F|TM) would have to be enormous to bring the entire expression anywhere close to 1. But a coarse-tuned ↳ universe would not be more likely than a fine-tuned universe on theism being true or a multiverse existing. If anything, fine-tuning may be more likely on theism than coarse-tuning.[14] So P(C|TM)/P(F|TM) is close to 1 or far less than 1. In either case, multiplying this value with a miniscule *n*/*w* will result in an equally tiny or even smaller number than *n*/*w*. The product of these two terms gives us the ratio of the evidential impact of coarse-tuning to the impact of fine-tuning. Since this ratio is so small, there should be a tremendous difference between the epistemic effects of the discovery of coarse-tuning and the discovery of fine-tuning.[15]

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Notes

1 I argue for this in “The Nature and Desiderata of Epistemic Expertise” (draft).

2 All other things being equal, of course. Some beliefs are more important or cover more territory regarding reality, while other beliefs are trivial and inconsequential. As I mentioned, I will ignore this importance dimension for the purposes of this chapter, but that dimension could be incorporated by weighting beliefs by their importance. When we compare how much cash two individuals have, we take the number of coins and bills they possess and weight them by their respective monetary values. If we are dealing with only coins that all have the same monetary value, e.g., pennies, then simply comparing the number of coins two individuals have will give us the right answer.

3 Even if the correct view of mental representation does not entail finite minds having infinitely many beliefs, Treanor could begin his argument using non-omniscient non-human beings that know an infinite number of truths, such as angels, and then rely on the ignorance version of his argument for finite human knowledge.

4 Patrick Grim has argued against the existence of a set of all truths (1984; 1991), appealing to Cantor’s Theorem, which states that the powerset $P(S)$ of a set S has a greater cardinality than S. Any candidate for the set of all sets will then be smaller than its powerset, and so will not be the set of all sets. Since there are truths associated with each set (such as set S not being an electron), Grim argues there cannot be a set of all truths. Different responses have been put forward to Grim’s claim. One approach is to use other axiomatizations of set theory, such as Quine’s New Foundations, which allow for a set of all sets. As Grim notes, they have some undesirable mathematical consequences. But there may be still other (yet unconceived) axiomatizations that allow for a set of all sets and do not have these negative consequences. Whichever way that debate turns out, I do not have to rely on a set of all truths for my purposes here; it is Treanor’s reductio that utilizes the notion of a set of all the truths. For the sake of argument, I will proceed with the assumption that we can somehow coherently talk about the set of all truths. But even if it turns out that there cannot be a set of all truths, Treanor could modify his reductio so that instead of utilizing a set of all truths, he uses a set of truths that is infinite while not including *all* the truths (for example, the union of all the truths believed by any human being who has ever lived together with an infinite set of arithmetic truths, such as whether each natural number is a prime or not).

5 Also, the total number of books we have taken out after an infinite number of such removals will be precisely the same as the number of books taken out during the first removal.

6 In the United States alone, the *Books In Print* database says that 304,912 new titles and editions were published in 2013. This does not include the estimated 1,108,183 self-published and on-demand books produced that year (http://www.bowker.com/news/2014/Traditional-Print-Book-Production-Dipped-Slightly-in-2013.html).

7 In correspondence, Treanor questions whether my book argument is parallel to his argument. He considers his T3 and T5 intuitively true, acceptable to anyone regardless of their views on infinity and cardinality, but he argues that my B3 and B5 require a non-Cantorian way of thinking about infinity. Naturally, I disagree. T3, B3, T5, and B5 are all expressions of pre-theoretic common-sense intuitions we have about how much knowledge and ignorance we have at different times and how many written and unwritten books there have been, as well as the relationships between these quantities. The conflicts between these pre-theoretic intuitions and the conclusions reached via Cantorian cardinality (T2 and B2) are needed to drive the arguments. If one considered T3 from a Cantorian perspective instead of the common-sense one, then they would reject T3, since as a good Cantorian, they would say that they are not any less ignorant than their ten-year-old self given the infinite cardinalities involved (recall that T1 supposes that cardinality is the measure of ignorance). So if my B3 is actually dependent on an explicitly non-Cantorian perspective as Treanor argues, his T3 is just as dependent on such a view (similar considerations apply to B5 and T5), preserving the symmetry between the two arguments.

8 It is inconsistent at least in sometimes giving us sets of differing cardinalities after removing from the same set two sets S1 and S2 that have the same cardinality, as in Craig’s library example. We expect that subtraction should always give the same-sized remainder when the cardinalities of the minuend and subtrahend do not change; it should not matter which members of a set are removed, only that the same number of members are removed. But these expectations are not satisfied in Cantorian set theory.

9 Matthew Parker discusses a number of historical figures, dating back to the medievals, who wrestled with these concepts (Parker 2009). Cantor put it this way: “the whole concept of number … in a certain sense *splits up* into *two* concepts when we ascend to the infinite” (Cantor [1883] 1976, 78, Cantor’s emphasis, quoted in Parker, 82). He was here thinking of his definition of cardinal numbers and what he termed ordinal numbers. Ordinal numbers can label elements of ordered sets and designate the length of such sets, and there are ordinal numbers that are between infinite cardinal numbers. For example, the ordering $<0, 1, 2, \dots>$ has an ordinal length of one less than the ordinal length of $<1, 2, 3, \dots, 0>$. Both include an infinite series of numbers, but only the second ordering has an additional item that follows after that series (Moore 2001, 123–7, 53). So Cantor recognized that for ordered sets at least, sets that had the same infinite cardinality could be different in another way of measuring their size. Ordinal size does not help us with the present epistemic problem,

however, since it is sensitive to rearrangement. As seen in the example above, the two orderings have the very same members, but the moving of 0 to the end makes their ordinal size different.

10 If it turns out that there cannot be a set of all truths (see n. 4), we could still talk about our ignorance decreasing by adopting the alternate approach I suggested in that note: using narrower but still infinitely large sets of truths instead of a set of all truths. For example, when I learn that 9791 is a prime number, there is one less truth that I am ignorant of in the infinite set composed of truths of the forms *n is a prime number* and *n is not a prime number*.

11 The square dartboard example is originally from Russell and Hawthorne (2016); similar examples are offered by Hájek (2003, 289), including one originally from Émile Borel, where the probability that a randomly chosen point on the Earth's surface lies on the equator is 0 and yet the probability that it lies in the western hemisphere is 0.5 given that it lies on the equator.

12 Thanks to Alan Hájek for informing me of the existence of that project.

13 This tiny conditional probability also tells us that the absolute probability of the constant falling in R_N with an infinite comparison range has to be tiny: it has to be less than or equal to n/w, since the absolute probability of the constant falling in R_W, whatever it is, cannot be greater than 1.

14 My paper "Preferring Stringent Laws of Nature: A Response to Weisberg on Fine Tuning" (draft) offers several reasons why God might prefer physical laws that require fine-tuning for life over laws that are not at all stringent in that sense.

Note that even if coarse-tuning were somehow more likely on TM than fine-tuning, it seems very unlikely that it would be so much more likely than fine-tuning as to counterbalance how tiny n/w is. The product of the likelihood ratios would still be far less than 1, which is all we need to show that coarse-tuning and fine-tuning should have different epistemic impacts.

15 Thanks to Alvin Plantinga, Robert Audi, and Patricia Blanchette for careful readings and discussions of portions of this chapter. The New Insights reading group offered helpful suggestions, especially Cameron Domenico Kirk-Giannini, Matthew Benton, and Max Baker-Hytch. Thanks to two anonymous reviewers and Nick Treanor for their comments. Finally, thanks to Jeffrey Sanford Russell for discussions regarding his square dartboard example. This publication was also made possible in part through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the author and do not necessarily reflect the views of the John Templeton Foundation.